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Classical Cut-elimination in the π-calculus

(In memory of Kohei Honda)

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Abstract We define the calculus $\mathcal{LK}$ - a variant of the calculus $\mathcal{X}$ - that enjoys the Curry-Howard correspondence for Gentzen's calculus LK; the variant consists of allowing arbitrary progress of *cut* over *cut*. We study the π-calculus enriched with pairing, for which we define a notion of implicative type assignment. We translate the terms of $\mathcal{LK}$ into this variant of π, and show that reduction and assignable types are preserved. This implies that all proofs in LK have a representation in π, and that *cut*-elimination is effectively simulated by π's synchronisation, congruence, and bisimilarity between processes.

We present two interpretations for which we show soundness results (but with respect to different notions of reduction), as well as type preservation. Using the second interpretation, we show that we preserve Gentzen's *Hauptsatz* result, and prove completeness.

We then enrich the logic with the connector $\neg$ (negation), and show that this also can be represented in π, whilst preserving the results.

Keywords classical logic, sequent calculus, pi calculus, translation, type assignment

Introduction

In this paper we present two translations of proofs of Gentzen's (implicative) proof calculus for Classical Logic LK [27] into the π-calculus [41] that respect *cut*-elimination. These translations are attained through using the intuition of the calculus $\mathcal{X}$, which gives a computational content to LK (a first version of this calculus was proposed in [45,47,46]; the implicative fragment of $\mathcal{X}$ was studied in [9]). We will here use a variant of $\mathcal{X}$, called $\mathcal{LK}$ – obtained by not using $\mathcal{X}$'s activated cuts but allowing arbitrary *cut*-over-*cut* reduction –

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which satisfies most properties shown to hold for $\mathcal{X}$ (with the exception of strong normalisation, but this is as expected for any calculus that models full *cut*-elimination).

$\mathcal{LK}$ enjoys the Curry-Howard isomorphism for LK, which it achieves by inhabiting the inference rules with term information, constructing witnesses for derivable sequents. Terms in $\mathcal{LK}$ are different from those in other calculi used for logic in that they have multiple named inputs and multiple named outputs, that are collectively called *connectors*. Reduction in $\mathcal{LK}$ is expressed via a set of rewrite rules that represent/correspond to *cut*-elimination in LK; reducing a term using these rules eventually leads to renaming of connectors and gives computational meaning to classical (sequent) proof reduction. It is well known that *cut*-elimination in LK is not confluent, and, since $\mathcal{LK}$ is Curry-Howard for LK and its reduction respects *cut*-elimination, neither is reduction in $\mathcal{LK}$.

These two main features of $\mathcal{X}$ –non-confluence and reduction as (re-)connection of terms via the exchange of names– are also manifest in the π-calculus, an observation which inspired us to consider the π-calculus as a means to model *cut*-elimination and proofs in LK. The aim of this paper is to link LK and π via $\mathcal{LK}$; we achieve this through the definition of two different translations that map (untyped) $\mathcal{LK}$-terms to π-processes: a natural translation that respects a notion of head reduction through synchronisation, and a semantic translation that respects weak bisimilarity in full. Although the origin of terms in $\mathcal{LK}$ are the proofs in LK, these translations in no way depend on type information, but map type-free terms (so also terms that do not correspond to proofs) to type-free processes. The translations focus, as is usual in semantics, on *observable behaviour*, and we will show that, if P reduces to Q, then the observable behaviour of Q is included in that of P[1] and that individual reduction steps are preserved.

However, we not only link LK and the π-calculus as systems of reduction, but also a systems of proofs. To that effect, we establish a relation between Classical Logic and the π-calculus by defining a notion of implicative type assignment for the latter, and show that the translations also preserve types, *i.e.* the image of a typeable term gives a process that is a witness to the same judgement; to achieve this result, the π-calculus is extended with pairing [2]. We thereby establish that the π-calculus has a strong link to functional languages with control and is thereby inherently more expressive than just the λ-calculus.

There are many more properties that one could demand to hold for these translations, like preservation of *compositions*, of *termination*, of *simulations*, of *equivalences*, *full abstraction*, etc. It is not immediately clear if checking these properties, or even aiming for them, makes sense in the context of the translations of $\mathcal{LK}$ we define here. After all, we are not interpreting one model of computation into another, but rather study the relation between cut-elimination in classical logic proofs and communication in a process calculus. The calculus $\mathcal{LK}$ we present here has not been proposed as a calculus to represent computation, is not a programming language and should not be treated as such, so it seems unreasonable to demand that the criteria we set on models of computation should also hold for $\mathcal{LK}$.

We will see that, for the kind of cut-elimination for LK as we consider in this paper, when allowing *cut*-over-*cut* reduction, *cut*-elimination is highly non-terminating, even looping for terms that intuitively should not; since for many terms that have a finite reduction path also a looping reduction path exists and our translations respect single reduction steps, we cannot hope to show that termination is preserved by our translations. However, this does not imply that *no* termination results can be shown; in fact, for the semantic translation we will show

[1] Since reduction is not confluent, it is not possible to show that P and Q have the same observable behaviour.

that Gentzen's *Hauptsatz* result (*i.e.* every provable judgement $\Gamma \vdash \Delta$ has a *cut*-free proof) is preserved.

Classical sequents

The *sequent calculus* LK, introduced by Gentzen in [27], is a logical system in which the rules only introduce connectives (but on either side of a sequent), in contrast to *natural deduction* (also introduced in [27]) which uses rules that introduce or eliminate connectives in the logical formulae. Natural deduction normally derives statements with a single conclusion, whereas LK allows for multiple conclusions, deriving sequents of the form $A_1,\ldots,A_n \vdash B_1,\ldots,B_m$, where $A_1,\ldots,A_n$ is to be understood as $A_1 \wedge \ldots \wedge A_n$ and B_1, $\ldots, B_m$ is to be understood as $B_1 \vee \ldots \vee B_m$. Kleene's version G_3 [38], with implicit weakening and contraction, of Implicative LK has four rules: *axiom*, *left introduction* of the arrow, *right introduction*, and *cut*:

$$(Ax):\ \frac{}{\Gamma, A \vdash_{\text{LK}} A, \Delta} \qquad\qquad (cut):\ \frac{\Gamma \vdash_{\text{LK}} A, \Delta \quad \Gamma, A \vdash_{\text{LK}} \Delta}{\Gamma \vdash_{\text{LK}} \Delta}$$

$$(\Rightarrow R):\ \frac{\Gamma, A \vdash_{\text{LK}} B, \Delta}{\Gamma \vdash_{\text{LK}} A \Rightarrow B, \Delta} \qquad\qquad (\Rightarrow L):\ \frac{\Gamma \vdash_{\text{LK}} A, \Delta \quad \Gamma, B \vdash_{\text{LK}} \Delta}{\Gamma, A \Rightarrow B \vdash_{\text{LK}} \Delta}$$

Since LK has no elimination rules, the only way to eliminate a connective is to eliminate the whole formula in which it appears via an application of the (cut)-rule. Gentzen defined a procedure that eliminates all applications of the (cut)-rule from a proof of a sequent using an innermost strategy, defined via local reductions of the proof-tree, which has –with some discrepancies– the flavour of term rewriting [39] or the evaluation of explicit substitutions [18,1]. His *Hauptsatz* result expresses that this kind of proof reduction is normalising.

The calculus $\mathcal{LK}$[2] achieves a Curry-Howard isomorphism - first discovered for Combinatory Logic [26] - for the proofs in LK by constructing *witnesses* for derivable sequents. This is established by, similar to calculi like Parigot's $\lambda\mu$ [42] and Curien and Herbelin's $\overline{\lambda\mu\tilde{\mu}}$ [25], attaching Roman names to formulae in the left context, and Greek names to those on the right, and to associate syntactic structure to the rules. Names on the left can be seen as inputs to the term, and names to the right as outputs; since multiple formulae can appear on both sides, this implies that a term can not only have more than one input, but also more than one output. There are two kinds of names (connectors) in $\mathcal{LK}$: *sockets* (inputs, with Roman names) and *plugs* (outputs, with Greek names), that correspond to *variables* and *co-variables*, respectively, in [48], or to Parigot's λ and μ-variables (see also [25]).

In the construction of the witness, when in applying a rule, a premise or conclusion disappears from the sequent and the corresponding name gets bound in the term that is constructed; when a premise or conclusion gets created, a different free (often, but not necessarily, new) name is associated to it. For example, in the creation of the term for right-introduction of the arrow

$$\frac{P \ \because\ \Gamma, x{:}A \vdash \alpha{:}B, \Delta}{\widehat{x}P\widehat{\alpha}\cdot\beta \ \because\ \Gamma \vdash \beta{:}A \rightarrow B, \Delta}$$

the input x and the output α are bound, and β is free. This case is interesting in that it highlights a special feature of $\mathcal{LK}$, not found in other calculi, which is the *simultaneous*

[2] Since the main difference between $\mathcal{X}$ [8,9] and $\mathcal{LK}$ is in the reduction rules, the observations in this section are true also for $\mathcal{X}$.

binding of two free names. Since in $\mathcal{LK}$ a term P can have many inputs and outputs, it is unsound to consider P a function *per se*; however, fixing *one* input x and *one* output α, we can see P as a function 'from x to α'. We make this limited view of P available via the output β, thereby *exporting* via β that 'P can be used as a function from x to α'. The types given to the connectors confirm this view.

Gentzen's proof reductions by *cut*-elimination[3] become the fundamental principle of computation in $\mathcal{LK}$. *Cuts* in proofs are witnessed by $P\widehat{\alpha} \dagger \widehat{x}Q$ (called the *cut* of P and Q via α and x), and the reduction rules specify how to remove them: a term is in normal form if and only if it has no sub-term of this shape. The intuition behind reduction is: the *cut* $P\widehat{\alpha} \dagger \widehat{x}Q$ expresses the intention to connect all occurrences of α in P and x in Q, and reduction will realise this by either connecting all αs to all xs (if x does not exist in Q, P will disappear), or all xs to all αs (if α does not exist in P, Q will disappear). Note that reduction in $\mathcal{LK}$ is not confluent; for example, as suggested above, when P does not contain α and Q does not contain x, reducing $P\widehat{\alpha} \dagger \widehat{x}Q$ can lead to both P and Q, two different terms.

Capturing $\mathcal{LK}$ in π

$\mathcal{LK}$'s notion of multiple inputs and outputs is also found in π, and was the original inspiration for our research. Our aim is to find a natural and intuitive translation of LK-proofs in π, and to devise a notion of type assignment for π so that the types in $\mathcal{LK}$ are preserved in π via this translation. In this precise sense we view processes in π as giving an alternative (computational) meaning to proofs in classical logic. To achieve this goal, we had to define a notion of type assignment that uses the type constructor $\rightarrow$ for π, and this is one of the contributions of this paper; we managed this without having to linearise the calculus as done in [37].

Although the calculi $\mathcal{LK}$ and π are, of course, fundamentally different, the similarities go beyond the correspondence of inputs and output between terms in $\mathcal{LK}$ and processes in π. Like $\mathcal{LK}$, π is application free, and substitution only takes place on *channel names*, similar to the connector-renaming feature of $\mathcal{LK}$, so *cut*-elimination is similar to synchronisation. The only dissimilarity lies in the fact that $\mathcal{LK}$ has explicit duplication of terms through reduction rules, whereas in the π-calculus this can only be achieved through replication, effectively "flooding the system." A *cut* $P\widehat{\alpha} \dagger \widehat{x}Q$ in $\mathcal{LK}$ expresses two terms that need to be connected via α and x. If we model P and Q in π through $[\![\cdot]\!]$, then we obtain one process sending on α, and one receiving on x (we can link these via $\alpha(w).\overline{x}\langle w\rangle$). Since each output on α in $[\![P]\!]$ takes place only once, and $[\![Q]\!]$ might want to receive in more than one x, we need to replicate the sending; likewise, since each input x in $[\![Q]\!]$ takes place only once, and $[\![P]\!]$ might have more than one send operation on α, $[\![Q]\!]$ needs to be replicated. However, there is no notion of erasure in the π-calculus, so these replicated terms are always present during the running of a process, giving rise to non-termination; we investigated placing guards on our translation to block replication (as used by Milner [41]), but found that this severely hampers its efficiency as well as provable results.

As discussed above, when creating a witness for $(\Rightarrow\!R)$ (the term $\widehat{x}P\widehat{\alpha}\cdot\beta$, called an *export*), the exported interface of P is the functionality of 'receiving on x, sending on α', which is made available on β. When interpreting this behaviour in π, we are faced with a problem. It is clearly not sufficient to limit communication to the exchange of single names,

[3] In his original paper [27], Gentzen never considered progressing a *cut* over a *cut*. In that sense, reduction in $\mathcal{LK}$ as we consider it here is much more 'liberal'; this comes at the price of losing strong normalisation.

since then we would have to separately send x and α, breaking perhaps the exported functionality, and certainly disabling the possibility of assigning arrow types. We overcome this problem by sending out a pair of names, as in $\overline{a}\langle\langle v,d \rangle\rangle$. Similarly, when interpreting a witness for $(\Rightarrow L)$ (the term $P\widehat{\alpha}\,[x]\,\widehat{y}Q$, called an *import*), the term that is to be connected to x is ideally a function whose input will be connected to α, and its output to y. This means that we need to receive a pair of names over x.

Related work

Logic and computation

The relation between *logic* and *computation* hinges around the Curry-Howard isomorphism (also attributed to de Bruijn), which expresses the fact that, for certain calculi with a notion of types, there exists a corresponding logic such that it becomes possible to associate terms with proofs, thus linking the term's type to the proposition shown by the proof, and proof contractions become term reductions. This phenomenon was first discovered for Combinatory Logic [26], and played an important part in de Bruijn's Automath.[4]

Before Herbelin's PhD [32] and Urban's PhD [45], the study of the relation between computation, programming languages and logic has concentrated mainly on *natural deduction systems* (of course, exceptions exist [29,30]). In fact, these carry the predicate '*natural*' deservedly; in comparison with, for example, *sequent style systems*, natural deduction systems are easy to understand and reason about. This holds most strongly in the context of *non-classical* logics; for example, the Curry-Howard relation between *Intuitionistic Logic* and the *Lambda Calculus* with types – of which the basic system is formulated by:

$$(Ax):\ \frac{}{\Gamma,x{:}A \vdash x:A} \qquad (\to I):\ \frac{\Gamma,x{:}A \vdash M:B}{\Gamma \vdash \lambda x.M:A\to B} \qquad (\to E):\ \frac{\Gamma \vdash M:A\to B \quad \Gamma \vdash N:A}{\Gamma \vdash MN:B}$$

– is well studied and understood, and has resulted in a vast and well-investigated area of research, resulting in, amongst others, functional programming languages and much further to system F [28] and the Calculus of Constructions [23]. In fact, all these calculi are *applicative* in that abstraction and application (corresponding to arrow introduction and elimination) are the main constructors in the syntax.

The link between Classical Logic and continuations and control was first established by Griffin for the λ_{C}-Calculus [31] (where C stands for Felleisen's C operator). Not much later, Parigot presented his $\lambda\mu$-calculus [42], an approach for representing classical proofs via a natural deduction system in which there is one main conclusion that is being manipulated, and possibly several alternative ones; the corresponding logic is one with *focus*. The $\lambda\mu$-calculus is presented as an extension of the λ-calculus, by extending the syntax with two new constructs that act as witness to the rules that deal with *conflict* $(\bot)$:

$$(\bot):\ \frac{\Gamma \vdash M:A \mid \alpha{:}A,\Delta}{\Gamma \vdash [\alpha]M:\bot \mid \alpha{:}A,\Delta} \qquad (\mu):\ \frac{\Gamma \vdash M:\bot \mid \alpha{:}A,\Delta}{\Gamma \vdash \mu\alpha.M:A \mid \Delta}$$

It uses two disjoint sets of variables (Roman and Greek characters). The sequents typing terms are of the form $\Gamma \vdash A \mid \Delta$, marking the conclusion A as *active*. That control can be expressed in the lazy variant of $\lambda\mu$ was shown in [24].

[4] `http://www.win.tue.nl/automath`

The introduction-elimination approach is easy to understand and convenient to use, but is also rather restrictive: for example, the handling of negation is not as nicely balanced, as is the treatment of contradiction (for a detailed discussion, see [44]). This imbalance can be observed in the $\lambda\mu$-calculus: adding $\perp$ as pseudo-type (only negation, or $A \rightarrow \perp$, is expressed; $\perp \rightarrow A$ is not a type), the $\lambda\mu$-calculus corresponds to *minimal classical logic* [5].

Herbelin has studied the calculus $\overline{\lambda}\mu\tilde{\mu}$ as an extension of $\lambda\mu$ without *application*, which gives a fine-grained account of manipulation of sequents [32,25,33]. The relation between call-by-name and call-by-value in the fragment of LK with negation and conjunction is studied in Wadler's Dual Calculus [48]; as in calculi like $\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$, that calculus considers a logic with *active* formulae, so these calculi do not achieve a direct Curry-Howard isomorphism with LK. The relation between $\mathcal{X}$ and $\overline{\lambda}\mu\tilde{\mu}$ has been investigated in [8,9]; there it was shown that it is straightforward to map $\overline{\lambda}\mu\tilde{\mu}$-terms into $\mathcal{X}$ whilst preserving reduction, but that it is only partially possible to do the converse.

π-calculus and logic

In the past, there have been several investigations of translations from various calculi (or logics) into the π-calculus [41], starting with Milner's seminal paper, presenting his input-based translation of the λ-calculus [13] into the π-calculus, and showing that the translation of closed λ-terms respects *lazy* reduction to normal form up to substitution. Many papers have been published in that area; here we concentrate on a review of the literature on the relationship between logic and the π-calculus.

The original idea of giving a computational translation of the *cut* as a communication primitive that we propose in this paper is also used by Abramsky in [4]; that paper was more a philosophical exposition of ideas, rather than a detailed presentation of an encoding with proofs. Abramsky's ideas were taken further by Bellin and Scott [16] and later by Bruscali and Gugliemi [20,19]. On the relation between Girard's linear logic [29] and the π-calculus, Bellin and Scott [16] give a treatment of information flow in proof-nets; only a small fragment of Linear Logic was considered, and the translation between proofs and π-calculus was left rather implicit as also noted by [21].

To illustrate this, notice that [16] uses the standard syntax for the polyadic π-calculus

$$\mathsf{P}, \mathsf{Q} \ ::= \ \mathsf{0} \mid \mathsf{P} \mid \mathsf{Q} \mid \,!\mathsf{P} \mid (\nu a)\, \mathsf{P} \mid a(\vec{x}).\mathsf{P} \mid \overline{a}\langle\vec{c}\rangle.\mathsf{P}$$

similar to the one we use here (see Definition 15) but for the fact that for us output is not synchronous, and there the *let*-construct is not used. However, the encoding of a 'cut' in linear logic

$$\dfrac{\vdash x{:}A \otimes B, y{:}(A \otimes B)^{\perp} \qquad \dfrac{\vdash n{:}A, m{:}A^{\perp} \qquad \vdash z{:}B, w{:}B^{\perp}}{\vdash m{:}A^{\perp}, w{:}B^{\perp}, v{:}A \otimes B}}{\vdash x{:}A \otimes B, m{:}A^{\perp}, w{:}B^{\perp}}$$

i.e. the 'term' $x{:}A \otimes B, m{:}A^{\perp}, w{:}B^{\perp}$, gets translated in [16] into a 'language of proofs' which looks like:

$$Cut^{k}(I, \bigotimes_{v}^{n,z}(I,I)mwz)x, (m,w) = (\nu k)\big(I[k/y] \mid \bigotimes_{v}^{n,z}(I,I)mwz[k/v]\big)$$

where the terms Cut and I are (rather loosely) defined. Notice the use of arbitrary application of processes to channel names, and the operation of pairing; the authors of [16] do not specify how to relate this notation to the above syntax of processes they consider.

However, even if this relationship is made explicit, even then a different π-calculus is needed to make the encoding work. To clarify this point, consider the translation in the π-calculus of the term above, which according to the definition given in [16] becomes:

$$(\nu k)\big(x(a).\underline{k(a)} \mid (\nu nz)(\underline{\vec{k}(n,z).}\big(n(b).m(b) \mid z(b).w(b)\big))\big).$$

Although intended, no communication is possible in this term. We have underlined the desired communication which is impossible, as the arity of the channel k does not match. To overcome this kind of problem, Bellin and Scott would need the *let* -construct with use of pairs of names as we have introduced in this paper in Definition 15. Moreover, there is no relation between the interpreted terms and proofs stated in [16] in terms of logic, types, or provable statements; here, we make a clear link between interpreted proofs and the logic through our notion of type assignment for the π-calculus.

Honda and Laurent [34] studied a typed π-calculus and show that a specific form of polarised linear logic [40] and a typed version of the asynchronous π-calculus [37] are essentially different ways of presenting the same structure. In contrast, our translations are very natural and intuitive by interpreting the *cut* operationally as a synchronisation in the basic, untyped π-calculus. Honda, Yoshida, and Berger [37] study a relation between a typed (*i.e.* types are part of the syntax of a term) Call-by-Value $\lambda\mu$ and a linear π-calculus; the translation they present is type dependent, in that, for each term, there are different π-processes assigned, depending on the original type; this makes the translation quite cumbersome. That paper achieves a *full abstraction* result, but at the price of considering only an explicitly typed version of $\lambda\mu$, restricted to CBV, and then only the lazy version of that, since it is essentially based on Milner's translation, *i.e.* does not model reduction in the right-hand side of an application; expressiveness of the results is obtained by changing the reduction strategy of the π-calculus by allowing synchronisations also under guard and under replication. So the results of [37] and our paper cannot really be compared; we just remark that our translation is type-free, maps onto a version of the π-calculus that does not allow for reduction to take place under replication or guard, deals with untypeable terms as well, and our semantic translation deals with reduction in a sequent calculus.

An accurate and elegant result on the relationship between π-calculus and linear logic is achieved by Beffara [14, 15]. In particular, in [15] various mappings of the $\lambda\mu$-calculus with linear logic types are encoded into synchronous π-calculus with forwarders. Observe that $\lambda\mu$ encodes a 'natural deduction' style of reasoning, while in this paper we are considering a sequent calculus kind of reasoning for classical logic in general, and not just for the fragment of linear logic.

In [12], two of the authors presented a compositional *output*-based translation for the $\Lambda\mu$-calculus (a variant of $\lambda\mu$ with separate naming and μ-binding operations) extended with explicit substitution, into the π-calculus with pairing, and showed that this translation preserves single-step explicit head reduction with respect to contextual equivalence. Since $\Lambda\mu$ is a λ-calculus, where reduction is confluent, the result of [12] is only partial with respect to the results we present here.

A result on the relation between classical logic and the π-calculus has appeared as [22], but for the fact that there a relation is established between the $\overline{\lambda}\mu\tilde{\mu}$-calculus and the π-calculus; since the focus for the translation as defined there is termination, it preserves only outermost reduction, which does not get formally motivated as a significant restriction of (proof)-reduction. Also, since in that approach all communication takes place via channels named λ, μ and $\tilde{\mu}$, it is not immediately clear that a natural notion of type assignment exists for π so that also type assignment is preserved.

Main results

In this paper, we will show results for two interpretations of $\mathcal{LK}$ into the π-calculus, each with their own strengths and provable properties.

The first translation $[\![\cdot]\!]_{\mathrm{N}}$ we define here is called *natural* since it closely follows the nature and structure of proofs in LK, and is, in approach, closely related to Milner's encoding and the output-based spine translation of [11]. The results we will show for this translation are:

Soundness : (Theorems 28 and 56) If P reduces to Q using *head* reduction, then the observational behaviour of $[\![P]\!]_{\mathrm{N}}$ contains that of $[\![Q]\!]_{\mathrm{N}}$ (as we will see in Remark 12, given the non-confluent nature of both $\mathcal{LK}$ and the π-calculus, we cannot show that $[\![P]\!]_{\mathrm{N}} = [\![Q]\!]_{\mathrm{N}}$) and if all head-reduction paths from P contain an infinite number of $(\mathit{exp\text{-}imp})$ steps, then $[\![P]\!]_{\mathrm{N}}$ diverges.

Preservation of types : (Theorems 44 and 58) If P is a witness for the judgement $\Gamma \vdash \Delta$, then so is $[\![P]\!]_{\mathrm{N}}$, effectively showing that all proofs in LK have a representation in the π-calculus.

It is possible to show that these results lead to soundness and preservation result for the encoding of the λ-calculus into the π-calculus via the encoding $\ulcorner\cdot\rfloor^{\mathrm{S}}\cdot$, *i.e.* the spine translation defined in [11] (essentially built out of the interpretation of λ-terms into $\mathcal{LK}$ and $[\![\cdot]\!]_{\mathrm{N}}$), or for the encoding $\ulcorner\cdot\rfloor\cdot$ of $\lambda\mu$-terms of [12], but we consider that out of scope for this paper.

These results show that the natural translation is strong, but as translation of full *cut*-elimination it falls short: not all reductions are modelled, and the translation is not *complete* (see Example 31). The first is almost standard in the literature: for example, [41,43] can only model *lazy* reduction, and, as argued in [11], only *explicit lazy* reduction in a step-by-step fashion. That last paper only manages to model *spine*-reduction (also known as *head*-reduction).

We will show here that we can overcome this (normal) restriction by presenting a second translation, $[\![\cdot]\!]_{\mathrm{S}}$, that we call *semantical*; it interprets terms as infinite resources, and is capable of representing *cut*-elimination in full, albeit not through mimicking reduction, but through bisimilarity (hence the moniker "semantical"). It is a generalisation of the natural translation in that it treats the interaction between a term and a context not through an input over the translation of the latter, as the natural translation does. For this second translation, we will show:

Operational Soundness : (Theorems 35 and 56) If P reduces to Q in $\mathcal{LK}$'s full reduction, then the observational behaviour of $[\![P]\!]_{\mathrm{S}}$ contains that of $[\![Q]\!]_{\mathrm{S}}$, and if all reduction paths from P contain an infinite number of $(\mathit{exp\text{-}imp})$ steps, then $[\![P]\!]_{\mathrm{S}}$ diverges.

Preservation of types : (Theorems 45 and 59) If P is a witness for the judgement $\Gamma \vdash \Delta$, so is $[\![P]\!]_{\mathrm{S}}$.

Operational completeness : (Theorem 37) If $[\![P]\!]_{\mathrm{S}}$ can be executed, then so can P, and these executions are related via $[\![\cdot]\!]_{\mathrm{S}}$ by: if $[\![P]\!]_{\mathrm{S}} \rightarrow_{\pi} \mathsf{Q}$ then there exists $P' \in \mathcal{LK}$, R such that $\mathsf{Q} \rightarrow_{\pi}^{*} \mathsf{R}$, $!\,\mathsf{R} \approx [\![P']\!]_{\mathrm{S}}$, and $P \rightarrow^{*} P'$.

Preservation of typeable termination : (Corollary 41) If P is typeable, then $[\![P]\!]_{\mathrm{S}}$ is bisimilar to a process in normal form; we hereby emulate Gentzen's *Hauptsatz* result.

In [7], we first presented our results on the translation of $\mathcal{LK}$-terms into the π-calculus; that paper also presented the notion of type assignment as defined here, as well as a proof that type assignment and cut-elimination are preserved by the translation. Since some details of the translations differ, we repeat these results here, with all particulars of the proofs;

moreover, here we define head reduction '$\rightarrow_{\text{H}}$' for $\mathcal{LK}$, and show that the translation $[\![\cdot]\!]_{\text{N}}$ respects $\rightarrow_{\text{H}}$; we also add the semantic translation $[\![\cdot]\!]_{\text{S}}$ and show that this is faithful with respect to $\mathcal{LK}$'s full reduction. In addition to [7] (and [9]), we treat the connective $\neg$ as well.

Overview of this paper

In Section 1, we give the definition of (implicative) $\mathcal{LK}$, followed by the notion of type assignment which establishes the Curry-Howard isomorphism. In Section 2, we show how to rewrite $\mathcal{LK}$-terms, and show the relation with LK's *cut*-elimination. The π-calculus with pairing is presented in Section 3. Section 4 defines the natural translation $[\![\cdot]\!]_{\text{N}}$ of $\mathcal{LK}$-terms into π-processes that closely follows the intuition of $\mathcal{LK}$, and shows a soundness and type-preservation result. In Section 5 we will modify the natural translation to represent $\mathcal{LK}$'s reduction in *full*, via the semantic translation $[\![\cdot]\!]_{\text{S}}$ and show soundness, type-preservation, and completeness. We will use this translation to show that every typeable term corresponds to a process in normal form, which emulates Gentzen's *Hauptsatz* result. In Section 6, we define a notion of type assignment for the π-calculus, and show that, under the two interpretations, typeable terms translate to processes that are typeable in the same way. Then, in Section 7 we look at how to represent negation in $\mathcal{LK}$, and study the relation between that representation and reduction. We conclude by representing negation directly in π, and show that type assignment is preserved also here.

1 The calculus $\mathcal{LK}$

In this section and the next we will give the definition of $\mathcal{LK}$, a variant of the calculus $\mathcal{X}$ which has been proven to be a fine-grained implementation model for various well-known calculi [8], like the λ-calculus, $\lambda\mu$, and $\overline{\lambda}\mu\tilde{\mu}$. As discussed in the introduction, the calculus $\mathcal{LK}$ is linked to Gentzen's sequent calculus LK; the system we will consider in this section has only implication, no structural rules and a changed axiom. $\mathcal{LK}$ features two separate categories of 'connectors', *plugs* and *sockets*, that act as output and input channels, respectively, and is defined without any notion of substitution or application. For the sake of clarity, we will develop our results first just for the implicative fragment of LK; we will consider negation in Section 7.

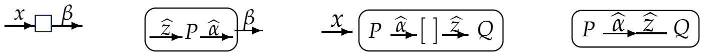
We would like to stress that the calculus $\mathcal{LK}$ we define here is not proposed as an abstract machine to model computation ($\mathcal{X}$ and $\overline{\lambda}\mu\tilde{\mu}$ are better suited for that), but just as a language that allows us to treat Gentzen's *cut*-elimination in a syntactical manner; we will see that reduction is highly inefficient, since looping on normalisable terms.

Definition 1 (Syntax) The terms of the $\mathcal{LK}$-calculus are defined by the following syntax, where the Roman characters x,y range over the infinite set of *sockets*, and the Greek characters α,β over the infinite set of *plugs*.

$$\begin{array}{rcll} P,Q & ::= & \langle x{\cdot}\beta\rangle & \textit{capsule} \\ & | & \widehat{z}P\widehat{\alpha}\cdot\beta & \textit{export} \\ & | & P\widehat{\alpha}\,[x]\,\widehat{z}Q & \textit{import} \\ & | & P\widehat{\alpha}\dagger\widehat{z}Q & \textit{cut} \end{array}$$

The $\hat{\cdot}$ symbolises that the socket or plug underneath is bound in the term.

We can represent these terms via the following diagrams (given just as a visual aid).



As an aid to intuition, ignoring the explicitly named outputs, we can see these terms with the view of other calculi:

$\mathcal{LK}$	$\lambda\mathbf{x}$	$\Lambda\mu$	$\overline{\lambda}\mu\tilde{\mu}$
$\langle x{\cdot}\beta\rangle$	x	$[\beta]x$	$\langle x \,|\, \beta\rangle$
$\widehat{z}P\widehat{\alpha}{\cdot}\beta$	$\lambda z.P$	$[\beta]\mu\alpha.\lambda z.P$	$\langle \lambda z.\mu\alpha.P \,|\, \beta\rangle$
$P\widehat{\alpha}\,[x]\,\widehat{z}Q$	$xPQ_1\cdots Q_n$	$[\omega](\mu\alpha.P)(\lambda v.[\omega]xv\lambda z.Q)$	$\langle x \,|\, \mu\alpha.P \cdot \tilde{\mu}z.Q\rangle$
$P\widehat{\alpha}\dagger\widehat{z}Q$	$Q\,\langle z := P\rangle$	$[\omega](\mu\alpha.P)(\lambda z.Q)$	$\langle \mu\alpha.P \,|\, \tilde{\mu}z.Q\rangle$

(where $\lambda\mathbf{x}$ is Bloo and Rose's λ-calculus with explicit substitution [17], and in the third case Q is seen as a context, acting as a stack of terms $Q_1, \ldots, Q_n$; for details, see [6] and [9]).

The encoding of the λ-calculus, $\lambda\mathbf{x}$, and $\lambda\mu$ into $\mathcal{LK}$ are defined in [9] through:

$$\begin{array}{rcl}
\lceil\!\lceil x \rfloor\!\rfloor_\alpha & \triangleq & \langle x{\cdot}\alpha\rangle \\
\lceil\!\lceil \lambda x.M \rfloor\!\rfloor_\alpha & \triangleq & \widehat{x}\lceil\!\lceil M \rfloor\!\rfloor_\beta\widehat{\beta}{\cdot}\alpha \\
\lceil\!\lceil MN \rfloor\!\rfloor_\alpha & \triangleq & \lceil\!\lceil M \rfloor\!\rfloor_\gamma\widehat{\gamma}\dagger\widehat{x}(\lceil\!\lceil N \rfloor\!\rfloor_\beta\widehat{\beta}\,[x]\,\widehat{y}\langle y{\cdot}\alpha\rangle) \\
\lceil\!\lceil M\,\langle x := N\rangle \rfloor\!\rfloor_\alpha & \triangleq & \lceil\!\lceil N \rfloor\!\rfloor_\beta\widehat{\beta}\dagger\widehat{x}\lceil\!\lceil M \rfloor\!\rfloor_\alpha \\
\lceil\!\lceil \mu\delta.[\gamma]M \rfloor\!\rfloor_\alpha & \triangleq & \lceil\!\lceil M \rfloor\!\rfloor_\gamma\widehat{\delta}\dagger\widehat{x}\langle x{\cdot}\alpha\rangle
\end{array}$$

Notice that terms are defined 'under output'. That paper also defines an interpretation of $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{LK}$:

$$\lceil\!\lceil \langle v \,|\, e\rangle \rfloor\!\rfloor \triangleq \lceil v \rfloor\!\rfloor_\alpha\widehat{\alpha}\dagger\widehat{x}\lceil e \rfloor\!\rfloor_x$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\lceil x \rfloor\!\rfloor_\alpha & \triangleq & \langle x{\cdot}\alpha\rangle & \lceil \alpha \rfloor\!\rfloor_x & \triangleq & \langle x{\cdot}\alpha\rangle \\
\lceil\!\lceil \lambda x.v \rfloor\!\rfloor_\alpha & \triangleq & \widehat{x}\lceil v \rfloor\!\rfloor_\beta\widehat{\beta}{\cdot}\alpha & \lceil v{\cdot}e \rfloor\!\rfloor_x & \triangleq & \lceil v \rfloor\!\rfloor_\alpha\widehat{\alpha}\,[x]\,\widehat{y}\lceil e \rfloor\!\rfloor_y \\
\lceil\!\lceil \mu\beta.c \rfloor\!\rfloor_\alpha & \triangleq & \lceil c \rfloor\!\rfloor\,\widehat{\beta}\dagger\widehat{x}\langle x{\cdot}\alpha\rangle & \lceil\!\lceil \tilde{\mu}y.c \rfloor\!\rfloor_x & \triangleq & \langle x{\cdot}\beta\rangle\widehat{\beta}\dagger\widehat{y}\lceil c \rfloor\!\rfloor
\end{array}$$

Here terms are interpreted under output, and contexts under input.

Definition 2 1. The *bound sockets* and *bound plugs* in a term are defined by:

$$\begin{array}{lcl}
bs\,(\langle x{\cdot}\alpha\rangle) & = & \emptyset \\
bs\,(\widehat{z}P\widehat{\alpha}{\cdot}\beta) & = & bs\,(P) \cup \{z\} \\
bs\,(P\widehat{\alpha}\,[x]\,\widehat{z}Q) & = & bs\,(P) \cup bs\,(Q) \cup \{z\} \\
bs\,(P\widehat{\alpha}\dagger\widehat{z}Q) & = & bs\,(P) \cup bs\,(Q) \cup \{z\}
\end{array}$$

$$\begin{array}{lcl}
bp\,(\langle x{\cdot}\alpha\rangle) & = & \emptyset \\
bp\,(\widehat{z}P\widehat{\alpha}{\cdot}\beta) & = & bp\,(P) \cup \{\alpha\} \\
bp\,(P\widehat{\alpha}\,[x]\,\widehat{z}Q) & = & bp\,(P) \cup \{\alpha\} \cup bp\,(Q) \\
bp\,(P\widehat{\alpha}\dagger\widehat{z}Q) & = & bp\,(P) \cup \{\alpha\} \cup bp\,(Q)
\end{array}$$

2. The set of *bound connectors* of P is defined by: $bc(P) = bs\,(P) \cup bp\,(P)$.
3. A socket x or plug α occurring in P which is not bound is called *free*, written $x \in fs(P)$ and $\alpha \in fp(P)$.

We will identify terms that only differ in the names of bound connectors, as usual, and write $x \notin fs(P,Q)$ for $x \notin fs(P)$ and $x \notin fs(Q)$, etc.

Notice that each term in $\mathcal{LK}$ has at least *one* free plug.

We accept Barendregt's convention on names, which states that no name can occur both free *and* bound in a context; α-conversion is supposed to take place silently, whenever necessary.

In order to come to a notion of type (or better: context) assignment for $\mathcal{LK}$, we define types and contexts.

Definition 3 (Types and Contexts)

1. The set of *(implicative) types* is defined by the grammar:

$$A, B \ ::= \ \varphi \mid A \rightarrow B$$

   where φ is a basic type of which there are countably many.[5]
2. A *context of sockets* Γ is a mapping from sockets to types, denoted as a finite set of *statements* $x{:}A$ such that the *subject* of the statements (x) are distinct. We write Γ_1, Γ_2 for the *compatible* union of Γ_1 and Γ_2 (if $x{:}A_1 \in \Gamma_1$ and $x{:}A_2 \in \Gamma_2$ then $A_1 = A_2$), and write $\Gamma, x{:}A$ for $\Gamma, \{x{:}A\}$.
3. Contexts of *plugs* Δ, and the notions Δ_1, Δ_2 and $\alpha{:}A, \Delta$ are defined in a similar way.

So, when writing a context as $\Gamma, x{:}A$, this implies that $x{:}A \in \Gamma$, or Γ is not defined on x.

The notion of type assignment on $\mathcal{LK}$ that we present in this section is Kleene's basic implicative system for Classical Logic (Gentzen's system LK) as described above. The Curry-Howard property is easily achieved by erasing all term-information.

Definition 4 (Typing for $\mathcal{LK}$)

1. *Type judgements*[6] for $\mathcal{LK}$ are expressed via a ternary relation $P : \Gamma \vdash_{\mathrm{LK}} \Delta$, where Γ is a context of *sockets* and Δ is a context of *plugs*, and P is a term. We say that P is the *witness* of this judgement.
2. *Type assignment for $\mathcal{LK}$* is defined by the following rules:

$$(cap): \frac{}{\langle x{\cdot}\beta\rangle \mathrel{:\cdot} \Gamma, x{:}A \vdash \beta{:}A, \Delta} \qquad (cut): \frac{P \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta \quad Q \mathrel{:\cdot} \Gamma, z{:}A \vdash \Delta}{P\widehat{\alpha} \dagger \widehat{z}Q \mathrel{:\cdot} \Gamma \vdash \Delta}$$

$$(exp): \frac{P \mathrel{:\cdot} \Gamma, z{:}A \vdash \alpha{:}B, \Delta}{\widehat{z}P\widehat{\alpha}{\cdot}\beta \mathrel{:\cdot} \Gamma \vdash \beta{:}A\rightarrow B, \Delta} \qquad (imp): \frac{P \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta \quad Q \mathrel{:\cdot} \Gamma, z{:}B \vdash \Delta}{P\widehat{\alpha}\,[x]\,\widehat{z}Q \mathrel{:\cdot} \Gamma, x{:}A\rightarrow B \vdash \Delta}$$

   We write $P \mathrel{:\cdot} \Gamma \vdash \Delta$ if there exists a derivation using these rules that has this judgement in the bottom line.

It is easy to show that weakening is admissible.

Notice that each term in $\mathcal{LK}$ has at least *one* free plug, so it is impossible to derive a statement like $P : \Gamma \vdash_{\mathrm{LK}} \emptyset$[7] and that Γ and Δ carry the types of the free connectors in P, as unordered sets. There is no notion of type for P itself, instead the derivable statement shows how P is connectable.

[5] These types are normally known as *natural* (or *Curry*) types.

[6] We use the notation of [9].

[7] This is possible in the extended system we will consider in Section 7.

Example 5 (A proof of Peirce's Law) The following is a proof for Peirce's Law in LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A,B}\,(Ax)}{\vdash A{\Rightarrow}B,A}\,({\Rightarrow}R) \qquad \dfrac{}{A \vdash A}\,(Ax)}{(A{\Rightarrow}B){\Rightarrow}A \vdash A}\,({\Rightarrow}L)}{\vdash ((A{\Rightarrow}B){\Rightarrow}A){\Rightarrow}A}\,({\Rightarrow}R)$$

and its inhabitation in $\mathcal{LK}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\langle y{\cdot}\delta\rangle \mathrel{:\cdot} y{:}A \vdash \delta{:}A, \eta{:}B}\,(cap)}{\widehat{y}\langle y{\cdot}\delta\rangle\widehat{\eta}\cdot\alpha \mathrel{:\cdot} \vdash \alpha{:}A{\to}B, \delta{:}A}\,(exp) \qquad \dfrac{}{\langle w{\cdot}\delta\rangle \mathrel{:\cdot} w{:}A \vdash \delta{:}A}\,(cap)}{(\widehat{y}\langle y{\cdot}\delta\rangle\widehat{\eta}\cdot\alpha)\widehat{\alpha}\,[z]\,\widehat{v}\langle v{\cdot}\delta\rangle \mathrel{:\cdot} z{:}(A{\to}B){\to}A \vdash \delta{:}A}\,(imp)}{\widehat{z}((\widehat{y}\langle y{\cdot}\delta\rangle\widehat{\eta}\cdot\alpha)\widehat{\alpha}\,[z]\,\widehat{v}\langle v{\cdot}\delta\rangle)\widehat{\delta}\cdot\gamma \mathrel{:\cdot} \vdash \gamma{:}((A{\to}B){\to}A){\to}A}\,(exp)$$

2 Reduction on $\mathcal{LK}$

The reduction rules for the calculus $\mathcal{LK}$ are directly inspired by the *cut*-elimination rules in LK. It is possible to define proof reduction in many ways; Gentzen decided to consider the simplest contractions, and considered only the last rule applied in the two sub-derivations of *cut*s:

$$\dfrac{\dfrac{\boxed{\qquad\quad}}{\Gamma \vdash_{\text{LK}} A,\Delta}\,(r) \qquad \dfrac{\boxed{\qquad\quad}}{\Gamma, A \vdash_{\text{LK}} \Delta}\,(l)}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

In case the formula A is introduced in both these sub-derivations (*i.e.* either $({\Rightarrow}R)$ and $({\Rightarrow}L)$, or (Ax) and $({\Rightarrow}L)$, or $({\Rightarrow}R)$ and (Ax), or (Ax) and (Ax)) the *cut* can be contracted directly; otherwise, a sub-proof gets 'pushed' into the one does not introduce the formula, one proof-step at the time; notice that this might apply to both, so a choice might have to be made, which in itself might lead to different results.

We model these proof-contraction steps via term rewriting rules for $\mathcal{LK}$. For example, since

$$\dfrac{\dfrac{\dfrac{\boxed{\mathcal{D}_1}}{\Gamma, A \vdash_{\text{LK}} B,\Delta}}{\Gamma \vdash_{\text{LK}} A{\Rightarrow}B,\Delta}\,({\Rightarrow}R) \qquad \dfrac{\dfrac{\boxed{\mathcal{D}_2}}{\Gamma \vdash_{\text{LK}} A,\Delta} \qquad \dfrac{\boxed{\mathcal{D}_3}}{\Gamma, B \vdash_{\text{LK}} \Delta}}{\Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta}\,({\Rightarrow}L)}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

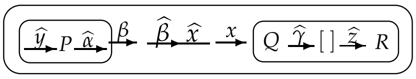
contracts to both:

$$\dfrac{\dfrac{\dfrac{\dfrac{\boxed{\mathcal{D}_2}}{\Gamma \vdash_{\text{LK}} A,\Delta}}{\Gamma \vdash_{\text{LK}} A,B,\Delta}\,(W) \qquad \dfrac{\boxed{\mathcal{D}_1}}{\Gamma, A \vdash_{\text{LK}} B,\Delta}}{\Gamma \vdash_{\text{LK}} B,\Delta}\,(cut) \qquad \dfrac{\boxed{\mathcal{D}_3}}{\Gamma, B \vdash_{\text{LK}} \Delta}}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

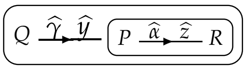
and

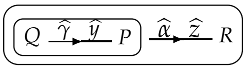
$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash_{\text{LK}} A,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \vdash_{\text{LK}} B,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_3\\ \Gamma, B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma, A, B \vdash_{\text{LK}} \Delta}\,(W)}{\Gamma, A \vdash_{\text{LK}} \Delta}\,(cut)}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

the witness for the first proof, $(\widehat{y}P\widehat{\alpha}\cdot\beta)\widehat{\beta} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$



reduces to both $Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{z}R)$ and $(Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\alpha} \dagger \widehat{z}R$





being the witnesses for the two resulting proofs; also this might lead to different results (*i.e.* *cut*-free proofs).

This behaviour is reflected in rule $(exp\text{-}imp)$, as presented in Definition 7. We can see the *cut* $(\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$ as a function $\widehat{y}P\widehat{\beta}\cdot\alpha$ (with body P, that takes input on y and outputs on β) interacting with a context $Q\widehat{\gamma}\,[x]\,\widehat{z}R$ (consisting of the function's argument Q, x as the hole that the function should occupy, and R the context of the 'explicit substitution' of P in Q). The contraction of the *cut* expresses (in the left-hand diagram) that the body of the function (which represents the result of the function, but with the substitution of the argument still pending) interacts with the context before using the argument; the other contraction first uses the argument, before interacting with the context, which corresponds to the standard way.[8]

Following Gentzen's approach, $\mathcal{LK}$'s term rules explain in detail how *cuts* are propagated through terms to be eventually evaluated at the level of *capsules*, where renaming takes place. Reduction is defined by specifying both the interaction between well-connected basic syntactic structures, and how to deal with propagating nodes to points in the term where they can interact. For this, it is important to know when a connector is introduced, *i.e.* is exposed and unique; informally, a term P introduces a socket x if P contains x and is constructed from sub-terms which do not contain x as free socket, so x only occurs at the 'top level.' This means that P is either an *import* with a middle connector $[x]$ or a *capsule* with left part x. Similarly, a term introduces a plug α if it is an *export* that 'creates' α or a *capsule* with right part α.

Definition 6 (Introduction)

P introduces α : Either $P = \widehat{x}Q\widehat{\beta}\cdot\alpha$ and $\alpha \notin fp(Q)$, or $P = \langle x\cdot\alpha\rangle$.
P introduces x : Either $P = Q\widehat{\beta}\,[x]\,\widehat{y}R$ with $x \notin fs(Q,R)$, or $P = \langle x\cdot\alpha\rangle$.

The logical reduction rules specify how to reduce a term that *cuts* sub-terms which introduce connectors. These rules are naturally divided in four categories: when a *capsule* is *cut* with a *capsule*, an *export* with a *capsule*, a *capsule* with an *import* or an *export* with an *import*. There is no other pattern in which a plug is introduced on the left of a '$\dagger$' and a socket is introduced on the right.

[8] In fact, in $\overline{\lambda}\mu\tilde{\mu}$ only the second alternative is represented.

Definition 7 (Logical rules) Let α and x be introduced in, respectively, the left and right-hand side of the main *cuts* below.

$$\begin{array}{rrcl}
(\mathit{cap}): & \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\beta\rangle & \to & \langle y{\cdot}\beta\rangle \\
(\mathit{exp}): & (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle & \to & \widehat{y}P\widehat{\beta}{\cdot}\gamma \\
(\mathit{imp}): & \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}R) & \to & Q\widehat{\beta}\,[y]\,\widehat{z}R \\
(\mathit{exp\text{-}imp}): & (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) & \to & \left\{\begin{array}{l} Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) \\ (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R \end{array}\right.
\end{array}$$

The first three logical rules above specify a renaming procedure, whereas the last rule specifies the basic computational step: it links the *export* of a function, available on the plug α, to an adjacent *import* via the socket x. The effect of the reduction will be that the exported function is placed in-between the two sub-terms of the *import*, acting as interface. Notice that two *cuts* are created in the result, that can be grouped in two ways; these alternatives do not necessarily have the same normal forms (since reduction is not confluent, normal forms are not unique).

We now define how to reduce a *cut* when one of its sub-terms does *not* introduce a connector mentioned in the *cut*. This will involve moving the *cut* inwards, towards a position where the connector *is* introduced. In case both connectors are not introduced, this search can start in either direction, giving another source of non-confluence.

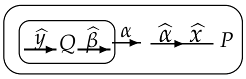
Similarly to the reasoning above, also the rules dealing with propagating *cuts* are inspired by Gentzen's *cut*-elimination rules. Take

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \vdash_{\text{LK}} A{\Rightarrow}B, B, \Delta\end{array}}{\Gamma \vdash_{\text{LK}} A{\Rightarrow}B, \Delta}\,(\Rightarrow R) \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma \vdash_{\text{LK}} \Delta}\,(\mathit{cut})$$

(notice the contraction towards $A{\Rightarrow}B$ in the left-hand sub-derivation, so the plug associated to this formula would not be introduced in the witness for $\Gamma \vdash_{\text{LK}} A{\Rightarrow}B, \Delta$) which reduces to

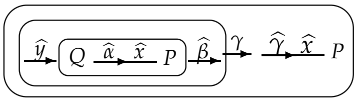
$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \vdash_{\text{LK}} A{\Rightarrow}B, B, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma, A \vdash_{\text{LK}} B, \Delta}\,(\mathit{cut})}{\Gamma \vdash_{\text{LK}} A{\Rightarrow}B, \Delta}\,(\Rightarrow R) \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma \vdash_{\text{LK}} \Delta}\,(\mathit{cut})$$

Notice that now in the conclusion of the left-hand sub-derivation, the formula $A{\Rightarrow}B$ is not contracted: therefore, in the witness for this proof, this is represented by an introduced plug; in fact, the witness for the first proof, the term $(\widehat{y}Q\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}P$, reduces to the witness for the second proof $(\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}{\cdot}\gamma)\widehat{\gamma} \dagger \widehat{x}P$ where now γ is introduced,[9] as reflected in rule $(\mathit{exp\text{-}outs}\dagger)$ below. So the diagram



with α free in Q, reduces to:

[9] We rename the outermost α to γ in order to adhere to Barendregt's convention.



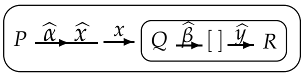
Also, since

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash_{\text{LK}} A{\Rightarrow}B,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, A{\Rightarrow}B \vdash_{\text{LK}} A,\Delta\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \Gamma, A{\Rightarrow}B, B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta}\,(\Rightarrow L)}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

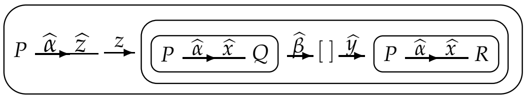
(again, notice the contraction) reduces to

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash_{\text{LK}} A{\Rightarrow}B,\Delta\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash_{\text{LK}} A{\Rightarrow}B,\Delta\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A{\Rightarrow}B \vdash_{\text{LK}} A,\Delta\end{array}}{\Gamma \vdash_{\text{LK}} A,\Delta}\,(cut) \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash_{\text{LK}} A{\Rightarrow}B,\Delta\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ \Gamma, A{\Rightarrow}B, B \vdash_{\text{LK}} \Delta\end{array}}{\Gamma, B \vdash_{\text{LK}} \Delta}\,(cut)}{\Gamma, A{\Rightarrow}B \vdash_{\text{LK}} \Delta}\,(\Rightarrow L)}{\Gamma \vdash_{\text{LK}} \Delta}\,(cut)$$

the term $P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{y}R)$ reduces to $P\widehat{\alpha} \dagger \widehat{z}((P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[z]\,\widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R))$, or:



(where x occurs free in Q or R) reduces to



as reflected in rule $(\dagger imp\text{-}outs)$.

This leads to the next set of rules that deal with cuts that do not have both connectors introduced, and define how to move that *cut* inwards.

Definition 8 (Propagation rules) Left propagation:

$$\begin{array}{rrcll}
(cap\dagger): & \langle y{\cdot}\beta\rangle\widehat{\alpha} \dagger \widehat{x}P & \to & \langle y{\cdot}\beta\rangle & (\beta \neq \alpha)\\
(exp\text{-}outs\dagger): & (\widehat{y}Q\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}P & \to & (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}{\cdot}\gamma)\widehat{\gamma} \dagger \widehat{x}P & \\
 & & & \multicolumn{2}{r}{(\gamma\textit{ fresh}, \alpha\textit{ not introduced})}\\
(exp\text{-}ins\dagger): & (\widehat{y}Q\widehat{\beta}{\cdot}\gamma)\widehat{\alpha} \dagger \widehat{x}P & \to & \widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}{\cdot}\gamma & (\gamma \neq \alpha)\\
(imp\dagger): & (Q\widehat{\beta}\,[z]\,\widehat{y}R)\widehat{\alpha} \dagger \widehat{x}P & \to & (Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\,[z]\,\widehat{y}(R\widehat{\alpha} \dagger \widehat{x}P) & \\
(cut\dagger): & (Q\widehat{\beta} \dagger \widehat{y}R)\widehat{\alpha} \dagger \widehat{x}P & \to & (Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta} \dagger \widehat{y}(R\widehat{\alpha} \dagger \widehat{x}P) &
\end{array}$$

Right propagation:

$$\begin{array}{lllll}
(\dagger cap): & P\widehat{\alpha} \dagger \widehat{x}\langle y{\cdot}\beta\rangle & \to & \langle y{\cdot}\beta\rangle & (y \neq x)\\
(\dagger exp): & P\widehat{\alpha} \dagger \widehat{x}(\widehat{y}Q\widehat{\beta}{\cdot}\gamma) & \to & \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}{\cdot}\gamma & \\
(\dagger imp\text{-}outs): & P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{y}R) & \to & P\widehat{\alpha} \dagger \widehat{z}((P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[z]\,\widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R)), & \\
 & & & \multicolumn{2}{r}{(z\textit{ fresh}, x\textit{ not introduced})}\\
(\dagger imp\text{-}ins): & P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[z]\,\widehat{y}R) & \to & (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[z]\,\widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R) & (z \neq x)\\
(\dagger cut): & P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) & \to & (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R) &
\end{array}$$

$$\begin{array}{llll}
(\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \boldsymbol{\dagger} \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{x}R) & \rightarrow_{\mathcal{LK}} & (\dagger \mathit{imp\text{-}outs}) & \\
(\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{y}(((\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{u}Q)\widehat{\tau}\,[y]\,\widehat{x}((\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \boldsymbol{\dagger} \widehat{u}R)) & \rightarrow_{\mathcal{LK}} & (\dagger \mathit{cap}) & \\
(\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{y}(((\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \boldsymbol{\dagger} \widehat{u}Q)\widehat{\tau}\,[y]\,\widehat{x}R) & \rightarrow_{\mathcal{LK}} & (\mathit{exp}), =_{\alpha} & \\
(\widehat{v}\langle v\cdot\rho\rangle\widehat{\rho}\cdot\gamma)\widehat{\gamma} \dagger \widehat{y}((\widehat{z}P\widehat{\delta}\cdot\tau)\widehat{\tau}\,[y]\,\widehat{x}R) & \rightarrow_{\mathcal{LK}} & (\mathit{exp\text{-}imp}) & \\
(\widehat{z}P\widehat{\delta}\cdot\tau)\widehat{\tau} \boldsymbol{\dagger} \widehat{v}(\langle v\cdot\rho\rangle\widehat{\rho} \dagger \widehat{x}R) & \rightarrow_{\mathcal{LK}} & (\dagger \mathit{cut}) & \\
((\widehat{z}P\widehat{\delta}\cdot\tau)\widehat{\tau} \boldsymbol{\dagger} \widehat{v}\langle v\cdot\rho\rangle)\widehat{\rho} \dagger \widehat{x}((\widehat{z}P\widehat{\delta}\cdot\tau)\widehat{\tau} \boldsymbol{\dagger} \widehat{v}R) & \rightarrow_{\mathcal{LK}} & (\mathit{exp}, \dagger \mathit{cap}) & \\
(\widehat{z}P\widehat{\delta}\cdot\rho)\widehat{\rho} \boldsymbol{\dagger} \widehat{x}R & \rightarrow_{\mathcal{LK}} & (\mathit{exp}) & \widehat{z}P\widehat{\delta}\cdot\sigma
\end{array}$$

Fig. 1 Running $(\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{x}R)$ of Example 10.

Definition 9 (Reduction)

1. We write $\rightarrow^{*}_{\mathcal{LK}}$ for the reduction relation defined as the smallest pre-order that includes the logical and propagation rules, extended with the contextual rules[10]

$$P \rightarrow Q \;\Rightarrow\; \left\{ \begin{array}{lcl}
\widehat{x}P\widehat{\alpha}\cdot\beta & \rightarrow & \widehat{x}Q\widehat{\alpha}\cdot\beta \\
P\widehat{\alpha}\,[x]\,\widehat{y}R & \rightarrow & Q\widehat{\alpha}\,[x]\,\widehat{y}R \\
R\widehat{\alpha}\,[x]\,\widehat{y}P & \rightarrow & R\widehat{\alpha}\,[x]\,\widehat{y}Q \\
P\widehat{\alpha} \dagger \widehat{y}R & \rightarrow & Q\widehat{\alpha} \dagger \widehat{y}R \\
R\widehat{\alpha} \dagger \widehat{y}P & \rightarrow & R\widehat{\alpha} \dagger \widehat{y}Q
\end{array} \right.$$

2. We define the notion of *head* reduction, $\rightarrow_{\mathrm{H}}$, by excluding reductions in and toward *import*, via the *elimination* of the propagation rules that move into an *import* (*i.e.* $(\mathit{imp}\dagger)$, $(\dagger\mathit{imp\text{-}outs})$, and $(\dagger\mathit{imp\text{-}ins})$, as well as the second and third contextual rule).
3. We defined *innermost* reduction $\rightarrow_{\mathrm{I}}$ by allowing the rules to be applied only to *cut*s composed out of terms in normal form (*i.e.* that contain no *cut*s).
4. We write $P\uparrow$ (and say that P *diverges*) if all reduction paths starting from P contain an infinite number of $(\mathit{exp\text{-}imp})$ steps.

Notice that this notion of reduction has many critical pairs, making reduction highly non-confluent.

The main difference between this reduction and that of $\mathcal{X}$ is that, essentially, in $\mathcal{X}$ *activated* *cut*s $P\widehat{\alpha} \not\nearrow \widehat{x}Q$ and $P\widehat{\alpha} \nwarrow\!\!\!\backslash\, \widehat{x}Q$ are added to the syntax, and only those are allowed to propagate over non-activated *cut*s; this is crucial for the Strong Normalisation result as shown by Urban for a syntactic variant of $\mathcal{X}$. Instead, the rewriting we consider here corresponds more closely to *free* *cut*-elimination.

Example 10 Taking $P = \langle z\cdot\delta\rangle$, $Q = \langle u\cdot\tau\rangle$ and $R = \langle x\cdot\sigma\rangle$ (notice that then u is not introduced in $Q\widehat{\tau}\,[u]\,\widehat{x}R$), we can reduce $(\widehat{z}P\widehat{\delta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{x}R)$ as in Figure 1 (notice that we have marked the cut that gets contracted).

Unlike a similar notion for the λ-calculus, our notion of head reduction is not deterministic: notice that

$$(\widehat{y}((\widehat{v}P\widehat{\delta}\cdot\sigma)\widehat{\sigma} \dagger \widehat{z}\langle z\cdot\gamma\rangle)\widehat{\gamma}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\beta\rangle \;\rightarrow_{\mathrm{H}}\; \left\{ \begin{array}{l}
(\widehat{y}(\widehat{v}P\widehat{\delta}\cdot\gamma)\widehat{\gamma}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\beta\rangle \\
\widehat{y}((\widehat{v}P\widehat{\delta}\cdot\sigma)\widehat{\sigma} \dagger \widehat{z}\langle z\cdot\gamma\rangle)\widehat{\gamma}\cdot\beta
\end{array} \right.$$

so both cuts can be contracted under $\rightarrow_{\mathrm{H}}$.

[10] Reduction in $\mathcal{LK}$ is defined as a term rewriting system, where the contextual rules are normally left implicit; we mention them here because we define a restriction of reduction that also limits the contextual rules.

Notice that our *cut*-elimination is different from Gentzen's original (implicit) definition: he in fact did not consider a *cut*-over-*cut* step, and used *innermost* reduction for his *Hauptsatz* result (see Proposition 38).

The soundness result of type assignment with respect to reduction is stated as usual:

Theorem 11 (Witness Reduction for $\mathcal{LK}$ [9]) *If $P :\cdot\ \Gamma \vdash \Delta$, and $P \rightarrow^{*}_{\mathcal{LK}} Q$, then $Q :\cdot\ \Gamma \vdash \Delta$.*

Although the reduction rules of $\mathcal{LK}$ are different from those of $\mathcal{X}$, since activated *cut*s are no longer used, given that activated *cut*s are typed in the same way as normal *cut*s the proof is almost identical to that presented in [9], and we will omit it here.

Remark 12 (On non-confluence) We have already remarked that the reduction relation is not confluent. In fact, let P and Q be such that α is not free in P and x is not free in Q, then we can show both $P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow^{*}_{\mathcal{LK}} P$ and $P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow^{*}_{\mathcal{LK}} Q$. So, in particular, a term P can have more than one normal form. Now when interpreting a term through its set of normal forms via $\ulcorner\cdot\rrbracket_{\mathrm{NF}}$, it is easy to show that, if $P \rightarrow^{*}_{\mathcal{LK}} Q$, then $\ulcorner Q\rrbracket_{\mathrm{NF}} \subseteq \ulcorner P\rrbracket_{\mathrm{NF}}$; so picking one reduction from P can then exclude the reachability of some of the other normal forms, and the set of reachable normal forms decreases during reduction. Something similar also holds for our translations into the π-calculus: if $P \rightarrow^{*}_{\mathcal{LK}} Q$, then $\llbracket P\rrbracket$ has more observable behaviour than $\llbracket Q\rrbracket$, expressed via $\llbracket P\rrbracket \sqsupseteq_{\pi} \llbracket Q\rrbracket$.

In [9, 10] some basic properties are shown for $\mathcal{X}$, which essentially show that the calculus is well behaved, as well as the relation between $\mathcal{X}$ and a number of other calculi. These results are valid also for $\mathcal{LK}$, and motivate the formulation of admissible rules:

Lemma 13 (Garbage Collection and Renaming [10])

$$\begin{array}{llllllll}
(\dagger gc): & P\widehat{\alpha} \dagger \widehat{x}Q & \rightarrow_{\mathcal{LK}} & Q & \textit{if } x \notin \mathit{fs}(Q) & (\dagger ren): & \langle z{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}P \rightarrow_{\mathcal{LK}} & P[z/x] \\
(gc\dagger): & P\widehat{\alpha} \dagger \widehat{x}Q & \rightarrow_{\mathcal{LK}} & P & \textit{if } \alpha \notin \mathit{fp}(P) & (ren\dagger): & P\widehat{\beta} \dagger \widehat{z}\langle z{\cdot}\alpha\rangle \rightarrow_{\mathcal{LK}} & P[\alpha/\beta]
\end{array}$$

The activated cuts of the original presentation of $\mathcal{X}$ were introduced by Urban with the main purpose of giving enough control over *cut*-elimination to prove strong normalisation, without sacrificing expressivity. The idea is that, once activated, a *cut* has to run to completion, and cannot be 'crossed' with another *cut*. Since we are here not dealing with strong normalisation, but with full proof contraction in LK, we do without activated cuts, but at the price of looping reduction.

Example 14 As an example of a looping reduction, take:

$$\begin{array}{llll}
P\widehat{\alpha} \boldsymbol{\dagger} \widehat{x}(\langle x{\cdot}\beta\rangle\widehat{\beta} \dagger \widehat{z}Q) & \rightarrow_{\mathcal{LK}} & (\dagger cut) & \\
(P\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\beta\rangle)\widehat{\beta} \dagger \widehat{z}(P\widehat{\alpha} \boldsymbol{\dagger} \widehat{x}Q) & \rightarrow_{\mathcal{LK}} & (\dagger gc) & (x \notin \mathit{fs}(Q)) \\
(P\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\beta\rangle)\widehat{\beta} \boldsymbol{\dagger} \widehat{z}Q & \rightarrow_{\mathcal{LK}} & (cut\dagger) & \\
(P\widehat{\beta} \boldsymbol{\dagger} \widehat{z}Q)\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle\widehat{\beta} \dagger \widehat{z}Q) & \rightarrow_{\mathcal{LK}} & (gc\dagger) & (\beta \notin \mathit{fp}(P)) \\
P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle\widehat{\beta} \dagger \widehat{z}Q) & & &
\end{array}$$

Moreover, assuming $P :\cdot\ \Gamma \vdash \alpha{:}A, \Delta$ and $Q :\cdot\ \Gamma, z{:}A \vdash \Delta$, we can construct a derivation for $P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle\widehat{\beta} \dagger \widehat{z}Q) :\cdot\ \Gamma \vdash \Delta$ and all the intermediate terms in the reduction above are typeable by the Witness Reduction result: so also *cut*-elimination does not terminate and typeability does not guarantee termination.

Notice that, following the innermost reduction path, this loop is immediately broken:

$$P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle\widehat{\beta} \boldsymbol{\dagger} \widehat{z}Q) \ \rightarrow_{\mathcal{LK}} (\dagger ren)\ \ P\widehat{\alpha} \dagger \widehat{x}Q[x/z] \ =_{\alpha}\ P\widehat{\alpha} \dagger \widehat{z}Q$$

We hazard a guess that this is why Gentzen considers only innermost reduction.

3 The asynchronous π-calculus with pairing

The notion of asynchronous π-calculus that we consider in this paper is different from the standard system defined by Honda and Tokoro in [35]. One reason for the change we make lies directly in the calculus that is going to be interpreted, $\mathcal{LK}$, in which a term can be constructed binding *two connectors simultaneously*. We will model function and context interaction into processes communication by sending *data* over channels, *i.e.* not just names, but also pairs of names, so, inspired by [2], add pairing: we introduce a structure over names, such that a channel may pass along not only names but also pairs of names (but not a pair of pairs). This does not imply that the calculus we consider is polyadic, however: always only *one* item can be sent, which is either a name of a pair of names. We also introduce the *let*-construct to deal with inputs of pairs of names that get distributed over the continuation.

To ease this definition, we deviate slightly from the normal practice, and write either Greek characters $\alpha, \beta, \upsilon, \ldots$ or Roman characters $x, y, z, \ldots$ for channel names; we use a, b, c, n for either a Greek or a Roman name.

The reason we use the asynchronous π-calculus rather than the normal synchronous variant will become clear after Definition 21; notice that this choice is not a restrictive one, since the asynchronous π-calculus is included in the synchronous one.

Definition 15 **($\pi_{\langle\rangle}$: the asynchronous π-calculus with pairing)**

1. *Channel names* and *data* are defined by:

$$\begin{array}{rcll} a,b,c,d & ::= & x \mid \alpha & \textit{names} \\ p & ::= & a \mid \langle a,b \rangle & \textit{data} \end{array}$$

Notice that pairing is *not* recursive.

2. *Processes* are defined by the grammar:

$$\begin{array}{rcll} P,Q & ::= & 0 & \textit{nil} \\ & \mid & P \mid Q & \textit{composition} \\ & \mid & !P & \textit{replication} \\ & \mid & (\nu a)\, P & \textit{restriction} \\ & \mid & a(x).P & \textit{input} \\ & \mid & \overline{a}\langle p \rangle & \textit{(asynchronous) output} \\ & \mid & \mathit{let}\, \langle x,y \rangle = p \ \mathit{in}\ P & \textit{let construct} \end{array}$$

3. We consider n bound in $(\nu n)\, P$, x bound in $a(x).P$, and x and y to be bound in the *let*-construct $\mathit{let}\, \langle x,y \rangle = p\ \mathit{in}\ P$. We call n free in P if it occurs in P and is not bound; we write $\mathit{fn}(P)$ for the set of free names in P, and write $\mathit{fn}(P,Q)$ for $\mathit{fn}(P) \cup \mathit{fn}(Q)$.
4. We abbreviate $a(x).\mathit{let}\, \langle y,z \rangle = x\ \mathit{in}\ P$ by $a(y,z).P$, and $(\nu m)\,(\nu n)\, P$ by $(\nu mn)\, P$, and write $\overline{a}\langle c,d \rangle$ rather than $\overline{a}\langle\langle c,d \rangle\rangle$.
5. We write $a \rightarrow b$ for the *forwarder* [36] $a(w).\overline{b}\langle w \rangle$ (called a *wire* in [43]).
6. A (process) context is simply a term with a hole $[\cdot]$.

Some remarks on the structure of processes should be made. Notice that data occurs only in two cases: $\overline{a}\langle p \rangle$ and $\mathit{let}\, \langle x,y \rangle = p\ \mathit{in}\ P$, and that then p is either a single name, or a pair of names. This implies that we do not allow $\langle a,b \rangle.P$, nor $a(\langle b,c \rangle).P$, nor $\overline{a}\langle\langle\langle b,c \rangle,d \rangle\rangle$, nor $(\nu\langle a,b \rangle)\, P$, nor $\mathit{let}\, \langle\langle a,b \rangle,y \rangle = p\ \mathit{in}\ P$, etc. Therefore substitution $P[p/x]$ is a partial operation, which depends on the places in P where x occurs.

Definition 16 (Congruence) The *structural congruence* is the smallest equivalence relation closed under contexts defined by the following rules:

$$\begin{array}{rclrcll}
P \mid 0 & \equiv & P & (\nu m)\,(\nu n)\,P & \equiv & (\nu n)\,(\nu m)\,P & \\
P \mid Q & \equiv & Q \mid P & (\nu n)\,(P \mid Q) & \equiv & P \mid (\nu n)\,Q & (n \notin fn(P)) \\
(P \mid Q) \mid R & \equiv & P \mid (Q \mid R) & !P & \equiv & P \mid !P \equiv !P \mid !P \equiv !!P & \\
(\nu n)\,0 & \equiv & 0 & \mathit{let}\,\langle x,y\rangle = \langle a,b\rangle\,\mathit{in}\,P & \equiv & P[a/x, b/y] &
\end{array}$$

Because of the last clause, we will not treat $\mathit{let}\,\langle x,y\rangle = \langle a,b\rangle\,\mathit{in}\,P$ as syntactic representation of a process; this implies, for example, that we do not deal explicitly with the process $\mathit{let}\,\langle x,y\rangle = \langle a,b\rangle\,\mathit{in}\,P$ in our type assignment system.

As usual, we will consider processes modulo congruence. Because of rule $(P \mid Q) \mid R \equiv P \mid (Q \mid R)$, we will normally not write brackets in a parallel composition of more than two processes.

Definition 17 (Reduction)

1. The *reduction relation* $\rightarrow_\pi$ over the processes of the $\pi_{\langle\rangle}$-calculus is defined by following (elementary) rules:

$$\begin{array}{rcll}
\overline{a}\langle b\rangle \mid a(x).Q & \rightarrow & Q[b/x] & \\
\overline{a}\langle b,c\rangle \mid a(x).Q & \rightarrow & Q[\langle a,b\rangle/x],\ \textit{if well defined} & \raisebox{1.5ex}{\textit{synchronisation}} \\
P \rightarrow Q' & \Rightarrow & (\nu n)\,P \rightarrow (\nu n)\,Q & \textit{binding} \\
P \rightarrow Q & \Rightarrow & (\nu n)\,P \rightarrow (\nu n)\,Q & \textit{binding} \\
P \rightarrow Q & \Rightarrow & P \mid R \rightarrow Q \mid R & \textit{composition} \\
P \equiv Q \ \&\ Q \rightarrow Q' \ \&\ Q' \equiv P' & \Rightarrow & P \rightarrow P' & \textit{congruence}
\end{array}$$

2. We write $\rightarrow_\pi^+$ for the transitive closure of $\rightarrow_\pi$, $\rightarrow_\pi^*$ for its reflexive and transitive closure; we write $\rightarrow_\pi (a)$ if we want to point out that a synchronisation took place over channel a, and write $(=_\alpha)$ if we want to point out that α-conversion has taken place.

   The following is easy to show.

Proposition 18 Let P, Q not contain a and $a \neq b$, then

$$\begin{array}{rcl}
(\nu a)\,(\overline{a}\langle b\rangle.P \mid a(x).Q) & \approx & P \mid Q[b/x] \\
(\nu a)\,(!\overline{a}\langle b\rangle.P \mid a(x).Q) & \approx & P \mid Q[b/x]
\end{array}$$

As remarked above, $Q[p/x]$ as used in the synchronisation rule needs to be well defined for the synchronisation to take place; this implies that, for example, a synchronisation like $\overline{a}\langle p\rangle \mid a(z).\overline{z}\langle c\rangle$ is stuck, as is $\overline{a}\langle p\rangle \mid a(z).\overline{b}\langle z,v\rangle$, etc. Note that

$$\begin{array}{rcl}
\overline{a}\langle b,c\rangle \mid a(x,y).P & \triangleq & \overline{a}\langle b,c\rangle \mid a(z).\mathit{let}\,\langle x,y\rangle = z\,\mathit{in}\,P \\
& \rightarrow_\pi & \mathit{let}\,\langle x,y\rangle = \langle b,c\rangle\,\mathit{in}\,P \\
& \equiv & P[b/x, c/y]
\end{array}$$

exactly as intended.

There are several notion of equivalence defined for the π-calculus: the one we consider here, and will show is related to our encodings, is that of weak bisimilarity.

Definition 19 (Weak bisimilarity)

1. We write $P\!\downarrow\!\overline{n}$ (and say that P *outputs on* n) if $P \equiv (\nu b_1 \ldots b_m)\,(\overline{n}\langle p\rangle \mid Q)$ for some Q, where $n \neq b_1 \ldots b_m$. We write $P\!\Downarrow\!\overline{n}$ (P *will output on* n) if there exists Q such that $P \rightarrow^*_\pi Q$ and $Q\!\downarrow\! n$.
   $P\!\downarrow\! n$ (P *inputs on* n) and $P\!\Downarrow\! n$ (P *will input on* n) are defined similarly.
2. A *barbed bisimilarity* $\stackrel{.}{\approx}$ is the largest symmetric relation such that $P \stackrel{.}{\approx} Q$ satisfies the following clauses:
   (a) if for each name n: if $P\!\downarrow\!\overline{n}$ then $Q\!\Downarrow\!\overline{n}$, and if $P\!\downarrow\! n$ then $Q\!\Downarrow\! n$;
   (b) for all P', if $P \rightarrow^*_\pi P'$, then there exists Q' such that $Q \rightarrow^*_\pi Q'$ and $P' \stackrel{.}{\approx} Q'$.
3. *Weak-bisimilarity* is the largest relation $\approx$ defined by: $P \approx Q$ if and only if $\mathsf{C}[P] \stackrel{.}{\approx} \mathsf{C}[Q]$ for any context $\mathsf{C}[\cdot]$.
4. We write $P \sqsubseteq_\pi Q$ if and only if there exists an R such that $Q \equiv P \mid R$, and write $P \underset{\approx}{\sqsubset}_\pi Q$ if and only if there exists R such that $R \approx Q$ and $P \sqsubseteq_\pi R$ (so $Q \approx P \mid R'$ for some R').

To illustrate $\underset{\approx}{\sqsubset}_\pi$, we can show that $P \underset{\approx}{\sqsubset}_\pi\, !P$ and $P \underset{\approx}{\sqsubset}_\pi P \mid Q$; clearly $\approx\, \subset\, \underset{\approx}{\sqsubset}_\pi$.

When we write $P \underset{\approx}{\sqsubset}_\pi Q$, we wish to express that that Q has more behaviour than P. Differently from a simulation definition, $\underset{\approx}{\sqsubset}_\pi$ determines exactly the extra behaviour in Q. This yield a clearer statement on the correctness of the encodings.

We will need the following property:

Lemma 20 Let a *be at most only used as output channel in* P *and as input channel in* Q *, then:* $(\nu a)\,(!P \mid !Q) \approx\, !(\nu a)\,(!P \mid !Q)$.

Proof (Sketch) We build the following symmetric relation:

$$\begin{array}{rcl}
\mathcal{R} & = & \{\langle(\nu a)\,(!P \mid !Q),\ (\nu a)\,(!P \mid !Q) \mid !(\nu a)\,(!P \mid !Q)\rangle\} \quad \cup \\
 & & \{\langle(\nu a)\,(P' \mid !P \mid !Q),\ (\nu a)\,(P' \mid !P \mid !Q) \mid !(\nu a)\,(!P \mid !Q)\rangle : \forall P', P \rightarrow^*_\pi P'\} \quad \cup \\
 & & \{\langle(\nu a)\,(Q' \mid !P \mid !Q),\ (\nu a)\,(Q' \mid !P \mid !Q) \mid !(\nu a)\,(!P \mid !Q)\rangle : \forall Q', P \rightarrow^*_\pi Q'\} \quad \cup \\
 & & \{\langle(\nu a)\,(R' \mid !P \mid !Q),\ (\nu a)\,(R' \mid !P \mid !Q) \mid !(\nu a)\,(!P \mid !Q)\rangle : \forall R', P \mid Q \rightarrow^*_\pi R'\}.
\end{array}$$

$\mathcal{R}$ is a a barbed bisimilarity, as the seemly apparent extra behaviour coming from the extra ! on the left-hand term is in fact already present in the term on the right-hand side. For exactly the same reason any context will not distinguish the terms in $\mathcal{R}$, hence $\mathcal{R}$ is a weak-bisimilarity.

4 A natural translation for $\mathcal{LK}$ into $\pi_{\langle\rangle}$ that respects head reduction

In this section we will present a first translation $[\![\cdot]\!]_{\mathrm{N}}$ of $\mathcal{LK}$ into $\pi_{\langle\rangle}$; it is called natural since it will create processes that output on names that are associated to plugs, and input on names that are associated to sockets, and tries as much as possible to encode the joining of connectors through substitution, following the syntactic structure of terms. Also, it is natural since we can show that head reduction $\mathcal{LK}$ is implemented through synchronisation in $\pi_{\langle\rangle}$ (see Theorem 28); for the semantic translation of the next section we can show that reduction in $\mathcal{LK}$ is represented through weak bisimilarity (see Theorem 35).

Our translation is based on the intuition formulated above: the ***cut*** $P\widehat{\alpha} \dagger \widehat{x}Q$ expresses the intention to connect all αs in P and xs in Q. Translated into $\pi_{\langle\rangle}$, this results in seeing P as trying to send at least as many times over α as Q is willing to receive over x, and Q trying to receive at least as many times over x as P is ready to send over α.

The main aim of the natural translation is to represent reduction, *i.e.* to be able to show that, if $P \rightarrow_{\mathcal{LK}} Q$, then $[\![P]\!]_{\mathrm{N}}$ and $[\![Q]\!]_{\mathrm{N}}$ are weakly bisimilar (which includes synchronisation over private channels). In view of the fact that some rules duplicate terms (as in $(\mathit{exp\text{-}outs}\dagger)$ and $(\dagger\mathit{imp\text{-}outs})$), we need to use replication; since the cut $P\widehat{\alpha} \dagger \widehat{x}Q$ can be seen as both the distribution of P into Q *and* Q into P, in the last alternative we need to replicate both components, which because of rule $(\mathit{exp\text{-}imp})$ forces us to use replication in the second and third case as well.

Although departing from $\mathcal{LK}$ it is natural to use Greek names for outputs and Roman names for inputs, by the very nature of the communication of the π-calculus (it is only possible to communicate using the *same* channel for in and output), in the implementation we are forced to use Greek names also for inputs, and Roman names for outputs.

We will first give the definition of the natural translation, and then explain the details.

Definition 21 (Natural translation of $\mathcal{LK}$ in $\pi_{\langle\rangle}$) The *natural* translation is defined by:

$$\begin{array}{rcl}
[\![\langle x{\cdot}\beta\rangle]\!]_{\mathrm{N}} & = & x(w).\overline{\beta}\langle w\rangle \\
[\![\widehat{z}P\widehat{\alpha}{\cdot}\beta]\!]_{\mathrm{N}} & = & (\nu z\alpha)\,(!\,[\![P]\!]_{\mathrm{N}} \mid \overline{\beta}\langle z,\alpha\rangle) \\
[\![P\widehat{\alpha}\,[x]\,\widehat{z}Q]\!]_{\mathrm{N}} & = & x(\alpha,z).(!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}}) \\
[\![P\widehat{\alpha} \dagger \widehat{z}Q]\!]_{\mathrm{N}} & = & (\nu z)\,(!\,[\![P[z/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}})
\end{array}$$

Let us investigate the intuition behind this definition for a moment. The interpretation of $P\widehat{\alpha} \dagger \widehat{z}Q$ will generate a process $[\![P]\!]_{\mathrm{N}}$ that (possibly) outputs on α, and $[\![Q]\!]$ that inputs on z; since the intention of the *cut* is that α and z are connected, we realise this directly by renaming α by z:[11]

$$[\![P\widehat{\alpha} \dagger \widehat{z}Q]\!]_{\mathrm{N}} \;=\; (\nu z)\,(!\,[\![P[z/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}})$$

Since z (and α) are bound in $P\widehat{\alpha} \dagger \widehat{z}Q$, z is restricted.

Likewise, when constructing the process that represents the term $P\widehat{\alpha}\,[x]\,\widehat{z}Q$, we will generate a process $[\![P]\!]_{\mathrm{N}}$ that outputs on α, and $[\![Q]\!]_{\mathrm{N}}$ that inputs on z. Notice that when that term is placed in a cut with an export

$$(\widehat{y}R\widehat{\beta}{\cdot}\gamma)\,\widehat{\gamma} \dagger \widehat{x}(P\widehat{\alpha}\,[x]\,\widehat{z}Q)$$

a reduction step can be made that generates the term $P\widehat{\alpha} \dagger \widehat{y}(R\widehat{\beta} \dagger \widehat{z}Q)$. Therefore, we can see the synchronisation over x as enabling the connection of α to y and β to z. So, in effect, then the synchronisation over x exchanges *two* names (hence the need for pairing), and we can see x in $[\![P\widehat{\alpha}\,[x]\,\widehat{z}Q]\!]_{\mathrm{N}}$ as an input that receives the names (here y and β) that will be connected to the names α and z; we can achieve that naturally 'in one go' by treating α and z as *variables*. We place the interpretations of P and Q in parallel and introduce a guard using x, that receives on x the channel names that are going to be substituted for α and z:

$$[\![P\widehat{\alpha}\,[x]\,\widehat{z}Q]\!]_{\mathrm{N}} \;=\; x(\alpha,z).(!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}})$$

This then causes the use of pairing in the interpretation of the export $\widehat{z}P\widehat{\alpha}{\cdot}\beta$ as well. When we see $[\![R]\!]_{\mathrm{N}}$ as a process that can input on z and output on α, then $[\![\widehat{z}P\widehat{\alpha}{\cdot}\beta]\!]_{\mathrm{N}}$ is the process that communicates that fact over β, by sending the two names:

$$[\![\widehat{z}P\widehat{\alpha}{\cdot}\beta]\!]_{\mathrm{N}} \;=\; (\nu z\alpha)\,(!\,[\![P]\!]_{\mathrm{N}} \mid \overline{\beta}\langle z,\alpha\rangle)$$

since z and α are bound in $\widehat{z}P\widehat{\alpha}{\cdot}\beta$, they are restricted.

[11] Notice that $(\nu z)\,(!\,[\![P[z/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}}) =_{\alpha} (\nu\alpha)\,(!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q[\alpha/z]]\!]_{\mathrm{N}})$ since z does not occur free in P and α not in Q; we will not distinguish these and swap between them whenever convenient.

Remark 22 We can now better illustrate why we need to use the asynchronous π-calculus. Take the term $(\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\gamma\rangle$, which by rule (exp) reduces to $\widehat{y}P\widehat{\beta}\cdot\gamma$. We now want the interpretation of the first term to at least include (in terms of $\underset{\sim}{\sqsupset}_\pi$) that of the second. Assume we would have defined

$$\llbracket \widehat{y}P\widehat{\beta}\cdot\alpha \rrbracket \;=\; (\nu y\beta)\,(\overline{\alpha}\langle y,\beta\rangle.!\llbracket P \rrbracket)$$

then we can only show:

$$\begin{array}{ll}
\llbracket (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\gamma\rangle \rrbracket & \triangleq \\
(\nu x)\,(!\,(\nu y\beta)\,(\overline{\alpha}\langle y,\beta\rangle.!\llbracket P \rrbracket) \mid !\,x(w).\overline{\gamma}\langle w\rangle) & \equiv, \triangleq \\
(\nu x)\,((\nu y\beta)\,(\overline{x}\langle y,\beta\rangle.!\llbracket P \rrbracket) \mid x(w).\overline{\gamma}\langle w\rangle) \mid \llbracket (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\gamma\rangle \rrbracket & \rightarrow_\pi (x) \\
(\nu y\beta)\,(!\llbracket P \rrbracket \mid \overline{\gamma}\langle y,\beta\rangle) \mid \llbracket (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x\cdot\gamma\rangle \rrbracket &
\end{array}$$

Notice that this last process then does not include $\llbracket \widehat{y}P\widehat{\beta}\cdot\gamma \rrbracket$; in fact, $(\nu y\beta)\,(\overline{\gamma}\langle y,\beta\rangle.!\llbracket P \rrbracket)$ and $(\nu y\beta)\,(!\llbracket P \rrbracket \mid \overline{\gamma}\langle y,\beta\rangle)$ are not even weakly bisimilar. So we are forced to place the output $\overline{\gamma}\langle y,\beta\rangle$ in parallel to the interpretation of P.

Remark 23 (On confluence and $\underset{\sim}{\sqsupset}_\pi$) As observed in Remark 12, the *cut* $P\widehat{\alpha} \dagger \widehat{x}Q$ – with α not in P and x not in Q – in $\mathcal{LK}$ runs via erasure to either P or Q, and reducing it decreases the set of reachable normal forms. Observe that in the image of $\mathcal{LK}$ in π, being built without using 'choice', there is no notion of *erasure* of processes; this implies that, using reduction in the π-calculus, we cannot model $P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow_{\mathcal{LK}} P$; we can at most show:

$$\llbracket P\widehat{\alpha} \dagger \widehat{x}Q \rrbracket_{\mathrm{N}} \;\triangleq\; (\nu x)\,(!\,\llbracket P[x/\alpha] \rrbracket_{\mathrm{N}} \mid !\,\llbracket Q \rrbracket_{\mathrm{N}}) \;\equiv\; !\,\llbracket P \rrbracket_{\mathrm{N}} \mid !\,\llbracket Q \rrbracket_{\mathrm{N}}$$

assuming $\alpha \notin fp(P)$ and $x \notin fs(Q)$. Now all reductions will take place in either $\llbracket P \rrbracket_{\mathrm{N}}$ or $\llbracket Q \rrbracket_{\mathrm{N}}$, and both parts will remain under reduction. This implies that, in this case, it is clear that the interpreted *cut* $\llbracket P\widehat{\alpha} \dagger \widehat{x}Q \rrbracket_{\mathrm{N}}$ must *contain* the behaviour of either its contractea, so, evidently, has more behaviour than both $\llbracket P \rrbracket_{\mathrm{N}}$ and $\llbracket Q \rrbracket_{\mathrm{N}}$ separately. As stated in Remark 12, this is natural for translations of non-confluent calculi, since there $P \rightarrow_{\mathcal{LK}} Q$ implies $\llbracket Q \rrbracket \subseteq \llbracket P \rrbracket$. We see this return in the formulation of the correctness result (Theorem 28) for the natural translation, which is formulated through the relation $\underset{\sim}{\sqsupset}_\pi$.

Since in this translation some sub-terms are placed under *input*, a full representation of reduction in $\mathcal{LK}$ cannot be achieved: it is not possible to reduce the (interpreted) terms that appear under an *input*. To accommodate for this shortcoming, to achieve a simulation result using this first translation, we have to restrict the notion of reduction on $\mathcal{LK}$ to that of *head* reduction. In view of the literature that exists on translations into the π-calculus, this is unfortunate but standard: the encoding forces a restriction on the modelled reduction rules. A similar limitation was already evident for Milner's encoding of the λ-calculus in [41], which manages only to show a preservation result for (large step) *lazy* reduction [3] for the λ-calculus, and is thereby also present in all research that is based on that approach; also [11] has to limit the notion of reduction to spine reduction, and [12] to head reduction.

As can be seen in Definition 21, *input* is used for the translation of *import*, so the restriction will consist of removing the rules that reduce under *import*; notice that this forces the exclusion of the second and third contextual rule, as well as the propagation rules $(imp\dagger)$, $(\dagger imp\text{-}outs)$, and $(\dagger imp\text{-}ins)$.

The choice for the terminology *head*-reduction can be motivated as follows. The only remaining reduction rules that deal with *imports* are:

$$\begin{array}{lrcl}
(\mathit{imp}): & \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}R) & \to & Q\widehat{\beta}\,[y]\,\widehat{z}R \\
(\mathit{exp\text{-}imp}): & (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) & \to & \begin{cases} Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) \\ (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R \end{cases}
\end{array}$$

The restriction we put on the rewriting system in head-reduction implies that we can only contract a *cut* $T\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$ if T is a term with α introduced; as observed above, we can compare this term, with discrepancies, to $TQ\overrightarrow{R_i}$ (where R is the context $[\,]\overrightarrow{R_i}$). In particular, under head-reduction, in the term $T\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$ all reduction takes place exclusively *inside* T (so in the head of the term $TQ\overrightarrow{R_i}$), and the *cut* mentioned explicitly will only be contracted after that reduction produces a term that introduces α in an *export*. Moreover, even when $T\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$ reduces to $(\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)$, we can continue running inside P. In any case, the contraction of this *cut* is postponed (for an introduced x; if x is not introduced, it will always be blocked, since propagation into an *import* is no longer allowed) until the head introduces α. Notice that head reduction in $\mathcal{LK}$ models more than just what we suggest here: continuing on the metaphor of the λ-calculus, the *cut* $P\widehat{\alpha} \dagger \widehat{x}Q$ corresponds to $Q\,\langle x := P\rangle$; head reduction allows reduction in both components of the *cut*, so allows for reduction *during* substitution.

Following on from these observations, it is clear that in terms of the representation of computable functions, head reduction is still fully expressive. In fact, the spine translation of [11] is a combination of the mapping of the λ-calculus into $\mathcal{X}$ (or of natural deduction into the sequent calculus) and the natural translation we define here. In that paper it is shown that explicit λ-spine reduction is preserved step-by-step by the induced combination of $\mathcal{LK}$'s head reduction and π's synchronisation; it also shows that typeability is preserved.

Remark 24 We can make the following observations:

- The synchronisations generated by the natural translation only involve processes of the shape:
$$x(w).\overline{\alpha}\langle w\rangle \qquad \overline{\beta}\langle x,\alpha\rangle \qquad z(\beta,y).(\mathsf{P} \mid \mathsf{Q})$$
so in particular, substitution $\mathsf{P}[p/x]$ is always well defined. These synchronisations are of the shape:
  - $(\nu x)\,((\nu y\beta)\,(\mathsf{P} \mid \overline{x}\langle y,\beta\rangle) \mid x(\alpha,z).(\mathsf{R} \mid \mathsf{Q})) \;\to_\pi\; (\nu y\beta)\,(\mathsf{P} \mid \mathsf{R}[y/\alpha] \mid \mathsf{Q}[\beta/z])$, and after the synchronisation over x, P can receive over y from $\mathsf{R}[y/\alpha]$ and send over β to $\mathsf{Q}[\beta/z]$; or
  - $(\nu x)\,((\nu y\beta)\,(\mathsf{P} \mid \overline{x}\langle y,\beta\rangle) \mid x(w).\overline{\alpha}\langle w\rangle) \;\to_\pi\; (\nu y\beta)\,(\mathsf{P} \mid \overline{\alpha}\langle y,\beta\rangle)$.
- All synchronisation takes place *only* over channels whose names are bound connectors in the terms that are interpreted. In particular,
  - no synchronisation is possible in $\llbracket P\widehat{\alpha} \dagger \widehat{x}Q\rrbracket_{\mathrm{N}} \triangleq (\nu x)\,(!\,\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})$ between and $\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}}$ and $\llbracket Q\rrbracket_{\mathrm{N}}$ but over channel x; and
  - no direct synchronisation is possible in $\llbracket P\widehat{\alpha}\,[x]\,\widehat{y}Q\rrbracket_{\mathrm{N}} \triangleq x(\alpha,y).(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})$ between $\llbracket P\rrbracket_{\mathrm{N}}$ and $\llbracket Q\rrbracket_{\mathrm{N}}$, even after input has been received over x.
- The translation is not trivial, since
$$\begin{array}{rcl}
\llbracket \widehat{y}(\widehat{z}\langle y{\cdot}\beta\rangle\widehat{\beta}{\cdot}\gamma)\widehat{\gamma}{\cdot}\alpha\rrbracket_{\mathrm{N}} & = & (\nu y\gamma)\,(!\,(\nu z\beta)\,(!y(w).\overline{\beta}\langle w\rangle \mid \overline{\gamma}\langle z,\beta\rangle) \mid \overline{\alpha}\langle y,\gamma\rangle) \\
\llbracket \widehat{x}\langle x{\cdot}\delta\rangle\widehat{\delta}{\cdot}\alpha\rrbracket_{\mathrm{N}} & = & (\nu x\delta)\,(!x(w).\overline{\delta}\langle w\rangle \mid \overline{\alpha}\langle x,\delta\rangle)
\end{array}$$
(witnesses of, respectively, $\vdash A{\to}B{\to}A$ and $\vdash C{\to}C$) yielding processes that differ under $\approx$.

As mentioned in the introduction, we added pairing to the π-calculus in order to be able to deal with arrow types. Notice that using the polyadic π-calculus instead would not be sufficient: since we would like the translation to respect reduction, in particular we need to be able to reduce the translation of $(\widehat{x}P\widehat{\alpha}\cdot\beta)\widehat{\beta} \dagger \widehat{z}\langle z\cdot\gamma\rangle$ to that of $\widehat{x}P\widehat{\alpha}\cdot\gamma$ (when β not free in P). So, choosing to interpret the *export* of x and α over β as $\overline{\beta}\langle x,\alpha\rangle$ would force the translation of $\langle z\cdot\gamma\rangle$ to always receive a pair of names. But requiring for the translation of a *capsule* to always deal with pairs of names is too restrictive: we will see that then only arrow types could be assigned, so it is desirable to allow those to deal with single names as well. So, rather than moving towards the polyadic π-calculus, we opt for letting communication send a single item, which is either a name or a pair of names.

Example 25 The translation of $\llbracket \widehat{z}((\widehat{y}\langle y\cdot\delta\rangle\,\widehat{\eta}\cdot\alpha)\widehat{\alpha}\;[z]\;\widehat{v}\langle v\cdot\delta\rangle)\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{N}}$, the witness of Peirce's law of Example 5, becomes:

$$(\nu z\delta)\,(z(\alpha,v).(!\,(\nu y\eta)\,(!y(w).\overline{\delta}\langle w\rangle \mid \overline{\alpha}\langle y,\eta\rangle)) \mid !v(w).\overline{\delta}\langle w\rangle) \mid \overline{\gamma}\langle z,\delta\rangle)$$

That this process is a witness of $\vdash ((A\to B)\to A)\to A$ is a straightforward application of Theorem 44.

The following is straightforward:

Proposition 26 (Free name preservation) $\alpha, x \notin fc(P)$, if and only if $\alpha, x \notin fn(\llbracket P \rrbracket_{\mathrm{N}})$.

We will show in Theorem 28 that we can mimic $\mathcal{LK}$'s head reduction in $\pi_{\langle\rangle}$: if $P \to_{\mathrm{H}} Q$, the image of the $\mathcal{LK}$-term P under the translation in $\pi_{\langle\rangle}$ reduces to some $\pi_{\langle\rangle}$-process that contains the behaviour of Q, but might have some extra behaviour as well. As will become clear also in the proofs below, this is in part due to the presence of replicated processes in the translation of the *cut*, but also comes from the fact that reduction in $\mathcal{LK}$ is not confluent, as discussed in Remark 23.

First we need to show the following:

Lemma 27 1. *Assume γ does not occur free in $P[\alpha/x]$, and x can only be a free* socket *in P, then:*

$$\begin{array}{l}(\nu\alpha)\,(!\,(\nu y\beta)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \quad \underset{\sim}{\sqsupset}_{\pi}\\ \quad (\nu y\beta)\,(!\,(\nu\alpha)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle)\end{array}$$

2. *Assume α can only be a* plug *in Q, and x can only be a* socket *in P, then:*

$$\begin{array}{l}(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \quad \approx\\ \quad (\nu\gamma)\,(!\,(\nu y\beta)\,(!\,(\nu x)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}})\end{array}$$

3. *Assume x can only be a* socket *in P, and α is only a* plug *in Q or R, then:*

$$\begin{array}{l}(\nu\alpha)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \quad \approx\\ \quad (\nu y)\,(!\,(\nu\alpha)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,(!\,\llbracket R\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}))\end{array}$$

Proof 1. Notice that γ and y do not occur in $\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}$. Therefore

$$\begin{array}{lll}(\nu\alpha)\,(!\,(\nu y\beta)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \underset{\sim}{\sqsupset}_{\pi} & \\ (\nu\alpha)\,((\nu y\beta)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \equiv & (\alpha \neq \gamma, y, \beta \notin \llbracket P[\alpha/x]\rrbracket_{\mathrm{N}})\\ (\nu y\beta)\,((\nu\alpha)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}})) \mid \overline{\gamma}\langle y,\beta\rangle & \approx & (20)\\ (\nu y\beta)\,(!\,(\nu\alpha)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle) & & \end{array}$$

2. Notice that x does not occur in Q, might occur as *socket* in P, and that α is a *plug* in Q; therefore x is only used for *output* in $\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}}$. We observe that

$$(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}})$$

and

$$(\nu\gamma)\,((\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle)) \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}})$$

are weakly bisimilar since the substitution of γ for x does not introduce any transition in the term ($\overline{x}\langle y,\beta\rangle$ and $\overline{\gamma}\langle y,\beta\rangle$ could communicate only with $!\,\llbracket P\rrbracket_{\mathrm{N}}$ and $!\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}}$ respectively) and is not restricting other transitions ($!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}}$ can only communicate with $!\,\llbracket P\rrbracket_{\mathrm{N}}$). Therefore:

$$\begin{array}{lll}
(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle)) \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) & \equiv & \\
(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle)) \mid !\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) & \approx & (=_{\alpha}, \gamma\ \mathit{fresh}) \\
(\nu\gamma)\,((\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle)) \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}}) & \approx & (\mathit{as\ in\ 1}) \\
(\nu\gamma)\,(!\,(\nu y\beta)\,(!\,(\nu x)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle)) \mid !\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}}) & &
\end{array}$$

3. Notice that α is not used for *input* in either $\llbracket R\rrbracket_{\mathrm{N}}$ or $\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}}$. The processes

$$(\nu\alpha)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}})$$

and

$$(\nu y)\,(!\,(\nu\alpha)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,(!\,\llbracket R\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}))$$

are weakly bisimilar for a reasoning similar to the one above. Therefore:

$$\begin{array}{ll}
(\nu\alpha)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \equiv \\
(\nu\alpha)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \approx \\
(\nu y)\,(!\,(\nu\alpha)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,(!\,\llbracket R\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}})) &
\end{array}$$

We can show the following correctness result for head reduction, $\rightarrow_{\mathrm{H}}$:

Theorem 28 (Operational Soundness of $\llbracket\cdot\rrbracket_{\mathrm{N}}$ with respect to $\rightarrow_{\mathrm{H}}$) *If $P \rightarrow_{\mathrm{H}}^{*} Q$, then there exists R such that $\llbracket P\rrbracket_{\mathrm{N}} \rightarrow_{\pi}^{*} \mathsf{R}$ with $\mathsf{R} \mathrel{\underset{\sim}{\sqsupset}}_{\pi} \llbracket Q\rrbracket_{\mathrm{N}}$.*

Proof By induction on the definition of reduction. We only show the more illustrative cases, and deal with the rules in the order they were presented in Section 2.

Logical rules: $(\mathit{cap}) : \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle \rightarrow \langle y{\cdot}\gamma\rangle :$

$$\begin{array}{lll}
\llbracket\langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}} & \triangleq & (\nu x)\,(!y(w).\overline{x}\langle w\rangle \mid !x(w).\overline{\gamma}\langle w\rangle) \ \approx\ !y(w).\overline{\gamma}\langle w\rangle \ \triangleq \\
!\,\llbracket\langle y{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}} & \mathrel{\underset{\sim}{\sqsupset}}_{\pi} & \llbracket\langle y{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}}
\end{array}$$

$(\mathit{exp}) : (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle \rightarrow \widehat{y}P\widehat{\beta}{\cdot}\gamma :$

$$\begin{array}{ll}
\llbracket(\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}} & \triangleq \\
(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !x(w).\overline{\gamma}\langle w\rangle) & \equiv \\
(\nu x)\,((\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid x(w).\overline{\gamma}\langle w\rangle \mid & \\
\qquad\qquad !\,(\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !x(w).\overline{\gamma}\langle w\rangle) & \rightarrow_{\pi} (x) \\
(\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) \mid \llbracket(\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}} & \triangleq \\
\llbracket\widehat{y}P\widehat{\beta}{\cdot}\gamma\rrbracket_{\mathrm{N}} \mid \llbracket(\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle\rrbracket_{\mathrm{N}} & \mathrel{\underset{\sim}{\sqsupset}}_{\pi} \llbracket\widehat{y}P\widehat{\beta}{\cdot}\gamma\rrbracket_{\mathrm{N}}
\end{array}$$

$(\mathit{imp}) : \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}R) \rightarrow Q\widehat{\beta}\,[y]\,\widehat{z}R\ :$

$$\begin{array}{lll}
\llbracket\langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}P)\rrbracket_{\mathrm{N}} & \triangleq & (\nu x)\,(!y(w).\overline{x}\langle w\rangle \mid !x(\beta,z).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) \ \approx \\
!y(\beta,z).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) & \triangleq & !\,\llbracket Q\widehat{\beta}\,[y]\,\widehat{z}R\rrbracket_{\mathrm{N}} \ \mathrel{\underset{\sim}{\sqsupset}}_{\pi}\ \llbracket Q\widehat{\beta}\,[y]\,\widehat{z}R\rrbracket_{\mathrm{N}}
\end{array}$$

$(\mathit{exp\text{-}imp}) : (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \rightarrow Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) :$

$$\begin{array}{lll}
\llbracket (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)\rrbracket_{\mathrm{N}} & \triangleq & \\
(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !\,x(\gamma,z).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) & \equiv, \triangleq & \\
(\nu x)\,((\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid x(\gamma,z).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid & & \\
\qquad\qquad \llbracket (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R)\rrbracket_{\mathrm{N}}) & \to_{\pi}(x), \equiv, \underset{\sim}{\sqsupset}_{\pi} & \\
(\nu y)\,(!\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid (\nu\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}})) & \approx & (20) \\
(\nu y)\,(!\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid !\,(\nu\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}})) & \triangleq & \\
(\nu y)\,(!\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\widehat{\beta} \dagger \widehat{z}R\rrbracket_{\mathrm{N}}) & \triangleq & \\
\llbracket Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R)\rrbracket_{\mathrm{N}} & &
\end{array}$$

For $(\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \to (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R$ the proof is similar, since

$$\begin{array}{ll}
(\nu y\beta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}}) & \equiv \\
(\nu\beta)\,((\nu y)\,(!\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}}) & \approx (20) \\
(\nu\beta)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\gamma]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}}) & \triangleq \\
(\nu\beta)\,(!\,\llbracket Q\widehat{\gamma} \dagger \widehat{y}P\rrbracket_{\mathrm{N}} \mid !\,\llbracket R[\beta/z]\rrbracket_{\mathrm{N}}) & \triangleq \ \llbracket (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R\rrbracket_{\mathrm{N}}
\end{array}$$

Left propagation: $(\mathit{cap}\dagger) : \langle y\cdot\beta\rangle\widehat{\alpha} \dagger \widehat{x}P \to \langle y\cdot\beta\rangle,\ \beta \neq \alpha$:

$$\begin{array}{l}
\llbracket \langle y\cdot\beta\rangle\widehat{\alpha} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} \quad \triangleq \quad (\nu x)\,(!\,\llbracket \langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \ \equiv \\
!\,\llbracket \langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}} \mid (\nu x)\,(!\,\llbracket P\rrbracket_{\mathrm{N}}) \ \underset{\sim}{\sqsupset}_{\pi} \ \llbracket \langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}}
\end{array}$$

$(\mathit{exp\text{-}outs}\dagger) : (\widehat{y}Q\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}P \to (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{x}P, \gamma\ \mathit{fresh}$:

$$\begin{array}{lll}
\llbracket (\widehat{y}Q\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} & \triangleq & \\
(\nu x)\,(!\,(\nu y\beta)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y,\beta\rangle) \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) & \approx & (272) \\
(\nu\gamma)\,(!\,(\nu y\beta)\,(!\,(\nu x)\,(!\,\llbracket Q[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\gamma/x]\rrbracket_{\mathrm{N}}) & \triangleq & \\
\llbracket (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\cdot\gamma)\widehat{\gamma} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} & &
\end{array}$$

$(\mathit{exp\text{-}ins}\dagger)$: $(\widehat{y}Q\widehat{\beta}\cdot\gamma)\widehat{\alpha} \dagger \widehat{x}P \to \widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\cdot\gamma, \gamma \neq \alpha.$

$$\begin{array}{ll}
\llbracket (\widehat{y}Q\widehat{\beta}\cdot\gamma)\widehat{\alpha} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} & \triangleq \\
(\nu\alpha)\,(!\,(\nu y\beta)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \underset{\sim}{\sqsupset}_{\pi} \ \ (271) \\
(\nu y\beta)\,(!\,(\nu\alpha)\,(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid \overline{\gamma}\langle y,\beta\rangle) & \triangleq \\
(\nu y\beta)\,(!\,\llbracket Q\widehat{\alpha} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} \mid \overline{\gamma}\langle y,\beta\rangle) & \triangleq \ \ \llbracket \widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\cdot\gamma\rrbracket_{\mathrm{N}}
\end{array}$$

$(\mathit{imp}\dagger)$: Excluded from $\to_{\mathrm{H}}$.

$(\mathit{cut}\dagger) : (Q\widehat{\beta} \dagger \widehat{y}R)\widehat{\alpha} \dagger \widehat{x}P \to (Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta} \dagger \widehat{y}(R\widehat{\alpha} \dagger \widehat{x}P)$:

$$\begin{array}{lll}
\llbracket (Q\widehat{\beta} \dagger \widehat{y}R)\widehat{\alpha} \dagger \widehat{x}P\rrbracket_{\mathrm{N}} & \triangleq & \\
(\nu\alpha)\,(!\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}) \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) & \approx & (273) \\
(\nu y)\,(!\,(\nu\alpha)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,(!\,\llbracket R\rrbracket_{\mathrm{N}} \mid !\,\llbracket P[\alpha/x]\rrbracket_{\mathrm{N}})) & \triangleq & \\
\llbracket (Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta} \dagger \widehat{y}(R\widehat{\alpha} \dagger \widehat{x}P)\rrbracket_{\mathrm{N}} & &
\end{array}$$

Right propagation: $(\dagger\mathit{cap}) : P\widehat{\alpha} \dagger \widehat{x}\langle y\cdot\beta\rangle \to \langle y\cdot\beta\rangle, y \neq x$:

$$\begin{array}{l}
\llbracket P\widehat{\alpha} \dagger \widehat{x}\langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}} \quad \triangleq \quad (\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket \langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}}) \ \equiv \\
(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}}) \mid !\,\llbracket \langle y\cdot\beta\rangle\rrbracket_{\mathrm{N}} \ \underset{\sim}{\sqsupset}_{\pi} \ \llbracket P\rrbracket_{\mathrm{N}}
\end{array}$$

$(\dagger\mathit{exp}) : P\widehat{\alpha} \dagger \widehat{x}(\widehat{y}Q\widehat{\beta}\cdot\gamma) \to \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\cdot\gamma.$: Like $(\mathit{exp\text{-}ins}\dagger)$.

$(\dagger\mathit{imp\text{-}outs}), (\dagger\mathit{imp\text{-}ins})$: Excluded from $\to_{\mathrm{H}}$.

$(\dagger\mathit{cut}) : P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) \to_{\mathcal{LK}} (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R)$:

$$\begin{array}{lll}
\llbracket P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R)\rrbracket_{\mathrm{N}} & \triangleq & \\
(\nu x)\,(!\,\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) & \approx & (20) \\
!\,(\nu x)\,(!\,\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,(\nu y)\,(!\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) & \approx & (273) \\
!\,(\nu y)\,(!\,(\nu x)\,(!\,\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q[y/\beta]\rrbracket_{\mathrm{N}}) \mid !\,(\nu x)\,(!\,\llbracket P[x/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) & \triangleq & \\
!\,\llbracket (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R)\rrbracket_{\mathrm{N}} & \underset{\sim}{\sqsupset}_{\pi} & \\
\llbracket (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R)\rrbracket_{\mathrm{N}} & &
\end{array}$$

$$\begin{array}{lll}
\llbracket P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle \widehat{\beta} \dagger \widehat{z}Q)\rrbracket_{\mathrm{N}} & \triangleq & \\
(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,(\nu z)\,(!\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})) & \approx & (\mathit{273}) \\
(\nu z)\,(!\,(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})) & \equiv & \alpha \notin \mathit{fp}(Q), z \notin \mathit{fs}(P) \\
(\nu z)\,(!\,(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}}) \mid !\,\llbracket Q\rrbracket_{\mathrm{N}}) \mid !\,(\nu\alpha)\,!\,\llbracket P\rrbracket_{\mathrm{N}} & \sqsupseteq_{\pi} & \\
(\nu z)\,(!\,(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}}) \mid !\,\llbracket Q\rrbracket_{\mathrm{N}}) & \triangleq & \\
\llbracket (P\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}z\rangle)\,\widehat{\beta} \dagger \widehat{z}Q\rrbracket_{\mathrm{N}} & \approx & (\mathit{27}) \\
(\nu\alpha)\,(!\,(\nu z)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}}) \mid !\,(\nu z)\,(!\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})) & \equiv & z \notin \mathit{fs}(P), \alpha \notin \mathit{fp}(Q) \\
!\,(\nu z)\,!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid (\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,(\nu z)\,(!\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})) & \sqsupseteq_{\pi} & \\
(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,(\nu z)\,(!\,\llbracket \langle\alpha{\cdot}z\rangle\,\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})) & \triangleq & \llbracket P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}z\rangle \widehat{\beta} \dagger \widehat{z}Q)\rrbracket_{\mathrm{N}}
\end{array}$$

Fig. 2 Translation of the looping reduction

Contextual rules: By induction. □

Notice that, in this proof, the only place where reduction plays a role is in the logical rules (exp) and $(\mathit{exp\text{-}imp})$. All other steps are dealt with by equivalence and replication. Moreover, notice that in the formulation of this result, through $\underset{\sim}{\sqsubset}_{\pi}$ essentially we remove a replication or remove a larger process in rules $(\mathit{cap}\dagger)$ and $(\dagger\mathit{cap})$; it is in the latter two cases that we restrict the set of reachable normal forms.

Moreover, since all reductions in the translation in the cases (exp) and $(\mathit{exp\text{-}imp})$ satisfy the condition of Proposition 18, the above result can be restated as:

Corollary 29 If $P \rightarrow_{\mathrm{H}}^{*} Q$, *then* $\llbracket P\rrbracket_{\mathrm{N}} \underset{\approx}{\sqsupset}_{\pi} \llbracket Q\rrbracket_{\mathrm{N}}$.

Example 30 Since all reduction steps in Example 14 are steps not involving *import*s, the interpretation of the first and last terms there are related via $\sqsupseteq_{\pi}$, and no reduction takes place in the simulation of the $\mathcal{LK}$-reduction, as illustrated in Figure 2. Notice that this shows that there exists R such that

$$\llbracket P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle \widehat{\beta} \dagger \widehat{z}Q)\rrbracket_{\mathrm{N}} \;\approx\; \llbracket P\widehat{\alpha} \dagger \widehat{x}(\langle x{\cdot}\beta\rangle \widehat{\beta} \dagger \widehat{z}Q)\rrbracket_{\mathrm{N}} \mid R$$

so there the processes ignored through $\underset{\approx}{\sqsupset}_{\pi}$ do not contribute to the observable behaviour.

Example 31 (On completeness) We cannot show that the interpretation is *complete*, since not all reductions in the image of the interpretation correspond to reduction in $\mathcal{LK}$. Consider the term $(\widehat{z}P\widehat{\delta}{\cdot}\gamma)\,\widehat{\gamma} \dagger \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{x}R)$, and assume that u is not introduced in $Q\widehat{\tau}\,[u]\,\widehat{x}R$. Observe that this term is in $\rightarrow_{\mathrm{H}}$-normal form.

However, notice that, since

$$\begin{array}{l}
\llbracket (\widehat{z}P\widehat{\delta}{\cdot}\gamma)\,\widehat{\gamma} \dagger \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{x}R)\rrbracket_{\mathrm{N}} \;\triangleq \\
\qquad (\nu u)\,(!\,(\nu z\delta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{u}\langle z,\delta\rangle) \mid !\,u(\tau,x).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}))
\end{array}$$

the translation builds a communication for the top-most *cut* $\gamma \dagger u$, which *can* run:

$$\begin{array}{l}
(\nu u)\,(!\,(\nu z\delta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{u}\langle z,\delta\rangle) \mid !\,u(\tau,x).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}})) \;\;\rightarrow_{\pi} (u) \\
(\nu u)\,(!\,(\nu z\delta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{u}\langle z,\delta\rangle) \mid \\
\qquad (\nu z\delta)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q[z/\tau]\rrbracket_{\mathrm{N}} \mid !\,\llbracket R[\delta/w]\rrbracket_{\mathrm{N}}) \mid \\
\qquad\qquad\qquad !\,u(\tau,x).(!\,\llbracket Q\rrbracket_{\mathrm{N}} \mid !\,\llbracket R\rrbracket_{\mathrm{N}}))
\end{array}$$

This is caused by the fact that, in the π-calculus the ‘connections’ between γ and u are all established ‘individually’, rather than all ‘in one go’ as they are in LK.

So in the translation we can perform synchronisations, whereas the translated term is in normal form, therefore we cannot show a completeness result for head reduction.

5 A semantic translation

In this section, we define a translation from terms in $\mathcal{LK}$ onto processes in $\pi_{\langle\rangle}$ that fully respects reduction in $\mathcal{LK}$, modulo bisimulation, as a variant of the natural translation presented above.

In the approach of $\llbracket\cdot\rrbracket_{\mathrm{N}}$, the *import* $P\widehat{\alpha}\,[x]\,\widehat{y}Q$ is expressed using $x(\alpha,y).(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid !\,\llbracket Q\rrbracket_{\mathrm{N}})$, where the plug α and the socket x become variables that will be replaced by, respectively, an *output* and *input* name of the communicating process. We will now modify that definition to make also reduction inside the translation of an *import* possible; for that, we need to revert to a previous version of the natural translation.

The original approach of [7] was, essentially following the translation of $\mathcal{X}$ into $\overline{\lambda}\mu\tilde{\mu}$, to interpret $P\widehat{\alpha}\,[x]\,\widehat{y}Q$ via

$$x(v,d).((\nu\alpha)\,(!\llbracket P\rrbracket \mid !\,\alpha{\rightarrow}v) \mid (\nu y)\,(!\,d{\rightarrow}y \mid !\llbracket Q\rrbracket))$$

so, rather than seeing α as an input variable, it sees α as the name of an output channel, and send its output to the input name that will be received in the variable v using the forwarder $\alpha{\rightarrow}v$; similarly, y is seen as an input channel, and the output from the channel which name will arrive in d is redirected to y via $d{\rightarrow}y$. Since output on α might be generated more than once inside $\llbracket P\rrbracket$, as well as input might be called for more than once on y inside $\llbracket Q\rrbracket$, both forwarders are replicated. However, using this approach the variables v and d appear *only* in the redirections, not in $\llbracket P\rrbracket$ or $\llbracket Q\rrbracket$, so these two processes appear unnecessarily under *input* in the translation. This is what the new translation $\llbracket\cdot\rrbracket_{\mathrm{S}}$ fixes: we build what we call a *communication cell* in $x(v,d).(!\,\alpha{\rightarrow}v \mid !\,d{\rightarrow}y)$, which deals with the redirections of the received mediator's interface, which we put in parallel with the translations of $\llbracket P\rrbracket_{\mathrm{S}}$ and $\llbracket Q\rrbracket_{\mathrm{S}}$.

We also choose to see terms as infinite resources rather than using replication to model substitution, so use inherent replication for all synchronisation. This is achieved by replicating *all* communication, *i.e.* all *input* and *output* actions. This replicated translation is easier to understand, but differs from the natural one in that it does not model reduction via reduction, but via bisimilarity (so does not really constitute an interpretation, but more a semantics), whereas the natural translation truly uses $\pi_{\langle\rangle}$'s reduction in the proofs.

We define:

Definition 32 (Semantic translation of $\mathcal{LK}$ into $\pi_{\langle\rangle}$)

$$\begin{array}{rcl}
\llbracket\langle x{\cdot}\alpha\rangle\rrbracket_{\mathrm{S}} & = & !\,x(w).\overline{\alpha}\langle w\rangle \\
\llbracket\widehat{y}Q\widehat{\beta}{\cdot}\alpha\rrbracket_{\mathrm{S}} & = & (\nu y\beta)\,(\llbracket Q\rrbracket_{\mathrm{S}} \mid !\,\overline{\alpha}\langle y,\beta\rangle) \\
\llbracket P\widehat{\alpha}\,[x]\,\widehat{y}Q\rrbracket_{\mathrm{S}} & = & (\nu\alpha y)\,(\llbracket P\rrbracket_{\mathrm{S}} \mid !\,x(v,d).(!\,\alpha{\rightarrow}v \mid !\,d{\rightarrow}y) \mid \llbracket Q\rrbracket_{\mathrm{S}}) \\
\llbracket P\widehat{\alpha} \dagger \widehat{x}Q\rrbracket_{\mathrm{S}} & = & (\nu\alpha x)\,(\llbracket P\rrbracket_{\mathrm{S}} \mid !\,\alpha{\rightarrow}x \mid \llbracket Q\rrbracket_{\mathrm{S}})
\end{array}$$

Notice that (for technical reasons) we also choose to use the forwarder in the translation of the *cut*, rather than using the renaming mechanism of Definition 21.

Remark 33 The encoding $\llbracket\cdot\rrbracket_{\mathrm{S}}$ generates a flat parallel composition of processes of the shape

$$!\,x(w).\overline{a}\langle w\rangle \qquad !\,\overline{\alpha}\langle y,\gamma\rangle \qquad !\,x(v,d).(!\,\alpha{\rightarrow}v \mid !\,d{\rightarrow}y) \qquad !\,\alpha{\rightarrow}x$$

where all channel names (so not the variables) are coming from the interpreted $\mathcal{LK}$-terms. Synchronisation over these channel names is possible only if generated by the interpretation of *cut*s.

For this translation, we can show that replication is implicit for encoded terms:

Lemma 34 $\llbracket P \rrbracket_{\mathrm{S}} \approx \,!\llbracket P \rrbracket_{\mathrm{S}}$.

Proof By induction on the structure of terms.

$$\begin{array}{l}
P = \langle x{\cdot}\alpha\rangle : \ \llbracket \langle x{\cdot}\alpha\rangle \rrbracket_{\mathrm{S}} \triangleq \,!x(w).\overline{\alpha}\langle w\rangle \equiv \,!!x(w).\overline{\alpha}\langle w\rangle \triangleq \,!\llbracket \langle x{\cdot}\alpha\rangle \rrbracket_{\mathrm{S}}. \\
P = \widehat{x}Q\widehat{\alpha}{\cdot}\beta : \ \ \llbracket \widehat{x}Q\widehat{\alpha}{\cdot}\beta \rrbracket_{\mathrm{S}} \triangleq (\nu x\alpha)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) \approx (IH)\ (\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) \\
\quad \approx (20)\ !(\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) \approx (IH)\ !(\nu x\alpha)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) \triangleq \,!\llbracket \widehat{x}Q\widehat{\alpha}{\cdot}\beta \rrbracket_{\mathrm{S}}
\end{array}$$

$$\begin{array}{lll}
P = Q\widehat{\alpha}\,[y]\,\widehat{x}R : \ \ \llbracket Q\widehat{\alpha}\,[y]\,\widehat{x}R \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\alpha{\to}v \mid !d{\to}x) \mid \llbracket R \rrbracket_{\mathrm{S}}) & \approx & (IH) \\
\quad (\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\alpha{\to}v \mid !d{\to}x) \mid !\llbracket R \rrbracket_{\mathrm{S}}) & \approx & (20) \\
\quad !(\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\alpha{\to}v \mid !d{\to}x) \mid !\llbracket R \rrbracket_{\mathrm{S}}) & \approx & (IH) \\
\quad !(\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\alpha{\to}v \mid !d{\to}x) \mid \llbracket R \rrbracket_{\mathrm{S}}) & \triangleq & !\llbracket Q\widehat{\alpha}\,[y]\,\widehat{x}R \rrbracket_{\mathrm{S}}
\end{array}$$

$$\begin{array}{lllll}
P = Q\widehat{\alpha} \dagger \widehat{x}R : \ \ \llbracket Q\widehat{\alpha} \dagger \widehat{x}R \rrbracket_{\mathrm{S}} & \triangleq & (\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket R \rrbracket_{\mathrm{S}}) & \approx & (IH) \\
\quad (\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid !\llbracket R \rrbracket_{\mathrm{S}}) & \approx (20) & !(\nu x\alpha)\ (!\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid !\llbracket R \rrbracket_{\mathrm{S}}) & \approx & (IH) \\
\quad !(\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket R \rrbracket_{\mathrm{S}}) & \triangleq & !\llbracket Q\widehat{\alpha} \dagger \widehat{x}R \rrbracket_{\mathrm{S}} & &
\end{array}$$

□

Notice that this lemma implies that we cannot model reduction in $\mathcal{LK}$ via synchronisation; however, as above (Theorem 28), we can show a preservation result for this translation modulo *weak bisimilarity*.

Theorem 35 (Operational soundness for $\llbracket \cdot \rrbracket_{\mathrm{S}}$ with respect to $\rightarrow_{\mathcal{LK}}$)
If $P \rightarrow^{*}_{\mathcal{LK}} Q$, then $\llbracket P \rrbracket_{\mathrm{S}} \underset{\sim}{\sqsupset}_{\pi} \llbracket Q \rrbracket_{\mathrm{S}}$.

Proof By induction on the definition of reduction in $\mathcal{LK}$: again we only show the interesting cases.

$$\begin{array}{lll}
\text{Logical rules: } (\mathit{exp}) : \ \ \llbracket (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}\langle x{\cdot}\gamma\rangle \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(\llbracket \widehat{y}P\widehat{\beta}{\cdot}\alpha \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket \langle x{\cdot}\gamma\rangle \rrbracket_{\mathrm{S}}) & \triangleq & \\
\quad (\nu\alpha x)\,((\nu y\beta)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\overline{\alpha}\langle y,\beta\rangle) \mid !\alpha{\to}x \mid !x(w).\overline{\gamma}\langle w\rangle) & \approx & (\alpha,x) \\
\quad (\nu y\beta)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\overline{\gamma}\langle y,\beta\rangle) & \triangleq & \llbracket \widehat{y}P\widehat{\beta}{\cdot}\gamma \rrbracket_{\mathrm{S}}
\end{array}$$

$$\begin{array}{lll}
(\mathit{imp}) \ : \ \ \llbracket \langle y{\cdot}\alpha\rangle\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}P) \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(!y(w).\overline{\alpha}\langle w\rangle \mid !\alpha{\to}x \mid (\nu\beta z)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !x(v,d).(!\beta{\to}v \mid !d{\to}z) \mid \llbracket P \rrbracket_{\mathrm{S}})) & \approx & (\alpha,x) \\
\quad (\nu\beta z)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\beta{\to}v \mid !d{\to}z) \mid \llbracket P \rrbracket_{\mathrm{S}}) & \triangleq & \llbracket Q\widehat{\beta}\,[y]\,\widehat{z}P \rrbracket_{\mathrm{S}}
\end{array}$$

$$\begin{array}{lll}
(\mathit{exp\text{-}imp}) \ : \ \ \llbracket (\widehat{y}P\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,((\nu y\beta)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\overline{\alpha}\langle y,\beta\rangle) \mid !\alpha{\to}x \mid & & \\
\qquad\qquad\qquad (\nu\gamma z)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !x(v,d).(!\gamma{\to}v \mid !d{\to}z) \mid \llbracket R \rrbracket_{\mathrm{S}})) & \approx & (\alpha,x) \\
\quad (\nu y\beta\gamma z)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid \llbracket Q \rrbracket_{\mathrm{S}} \mid !\,\gamma{\to}y \mid !\,\beta{\to}z \mid \llbracket R \rrbracket_{\mathrm{S}}) & \equiv & \\
\quad (\nu\gamma y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\gamma{\to}y \mid (\nu\beta z)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\beta{\to}z \mid \llbracket R \rrbracket_{\mathrm{S}})) & \triangleq & \llbracket Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) \rrbracket_{\mathrm{S}}
\end{array}$$

For the second alternative of this rule, the proof is similar.

$$\begin{array}{lll}
\text{Left propagation: } (\mathit{cap}\dagger) : \ \ \llbracket \langle y{\cdot}\beta\rangle\widehat{\alpha} \dagger \widehat{x}P \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(\llbracket \langle y{\cdot}\beta\rangle \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) & \equiv & (\beta \neq \alpha) \\
\quad \llbracket \langle y{\cdot}\beta\rangle \rrbracket_{\mathrm{S}} \mid (\nu\alpha x)\,(!\,\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) & \underset{\sim}{\sqsupset}_{\pi} & \llbracket \langle y{\cdot}\beta\rangle \rrbracket_{\mathrm{S}}
\end{array}$$

$$\begin{array}{lll}
(\mathit{exp\text{-}outs}\dagger) : \ \ \llbracket (\widehat{y}Q\widehat{\beta}{\cdot}\alpha)\widehat{\alpha} \dagger \widehat{x}P \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,((\nu y\beta)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\alpha}\langle y,\beta\rangle) \mid !\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) & \equiv,\approx & (34) \\
\quad (\nu\alpha x)\,((\nu y\beta)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\alpha}\langle y,\beta\rangle) \mid !\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) & =_{\alpha} & \\
\quad (\nu\gamma x)\,((\nu y\beta)\,((\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) \mid !\overline{\gamma}\langle y,\beta\rangle) \mid !\gamma{\to}x \mid \llbracket P \rrbracket_{\mathrm{S}}) & \triangleq & \\
\quad \llbracket (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}{\cdot}\gamma)\widehat{\gamma} \dagger \widehat{x}P \rrbracket_{\mathrm{S}} & &
\end{array}$$

$$
\begin{array}{lll}
(\mathit{imp}\dagger) : \quad \llbracket (Q\widehat{\beta}\,[z]\,\widehat{y}R)\widehat{\alpha} \dagger \widehat{x}P \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,((\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}}) \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}}) & \equiv,\approx & (34) \\
\quad (\nu\alpha x)\,((\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}}) \mid & & \\
\qquad\qquad\qquad\qquad {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}}) & =_{\alpha} & \\
\quad (\nu\beta y)\,((\nu\alpha x)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}}) \mid & & \\
\qquad\qquad {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid (\nu\alpha x)\,(\llbracket R \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}})) & \triangleq & \\
\quad (\nu\beta y)\,(\llbracket Q\widehat{\alpha} \dagger \widehat{x}P \rrbracket_{\mathrm{S}} \mid {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R\widehat{\alpha} \dagger \widehat{x}P \rrbracket_{\mathrm{S}}) & \triangleq & \\
\quad \llbracket (Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta}\,[z]\,\widehat{y}(R\widehat{\alpha} \dagger \widehat{x}P) \rrbracket_{\mathrm{S}} & &
\end{array}
$$

$$
\begin{array}{ll}
\text{Right propagation: } (\dagger\mathit{exp}) : \quad \llbracket P\widehat{\alpha} \dagger \widehat{x}(\widehat{y}Q\widehat{\beta}\cdot\gamma) \rrbracket_{\mathrm{S}} & \triangleq \\
\quad (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid (\nu y\beta)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!\overline{\gamma}\langle y,\beta\rangle})) & \equiv \\
\quad (\nu y\beta)\,((\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket Q \rrbracket_{\mathrm{S}}) \mid {!\overline{\gamma}\langle y,\beta\rangle}) & \triangleq \\
\quad (\nu y\beta)\,(\llbracket P\widehat{\alpha} \dagger \widehat{x}Q \rrbracket_{\mathrm{S}} \mid {!\overline{\gamma}\langle y,\beta\rangle}) & \triangleq \ \llbracket \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\cdot\gamma \rrbracket_{\mathrm{S}}
\end{array}
$$

$$
\begin{array}{lll}
(\dagger\mathit{imp\text{-}outs}) : \quad \llbracket P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{y}R) \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid (\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!x(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}})) & \equiv,\approx & (34) \\
\quad (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid & & \\
\qquad\qquad\qquad\qquad (\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!x(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}})) & =_{\alpha} & \\
\quad (\nu\alpha z)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow z} \mid (\nu\beta y)\,((\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket Q \rrbracket_{\mathrm{S}}) \mid & & \\
\qquad\qquad {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket R \rrbracket_{\mathrm{S}}))) & \triangleq & \\
\quad \llbracket P\widehat{\alpha} \dagger \widehat{z}((P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[z]\,\widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R)) \rrbracket_{\mathrm{S}} & &
\end{array}
$$

$$
\begin{array}{lll}
(\dagger\mathit{imp\text{-}ins}) : \quad \llbracket P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[z]\,\widehat{y}R) \rrbracket_{\mathrm{S}} & \triangleq & \\
\quad (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid (\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}})) & \equiv,\approx & (34) \\
\quad (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid & & \\
\qquad\qquad\qquad\qquad (\nu\beta y)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid \llbracket R \rrbracket_{\mathrm{S}})) & \equiv & \\
\quad (\nu\beta y)\,((\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket Q \rrbracket_{\mathrm{S}}) \mid & & \\
\qquad\qquad {!z(v,d).(!\beta\rightarrow v \mid !d\rightarrow y)} \mid (\nu\alpha x)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid {!\alpha\rightarrow x} \mid \llbracket R \rrbracket_{\mathrm{S}})) & \triangleq & \\
\quad \llbracket (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[z]\,\widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R) \rrbracket_{\mathrm{S}} & &
\end{array}
$$

The contextual rules, as well as the transitive closure, follow by induction. □

Notice that in this proof $\mathrel{\underset{\sim}{\sqsupset}}_{\pi}$ is only needed in part $(\mathit{cap}\dagger)$ and $(\dagger\mathit{cap})$, where we eliminate part of a process, otherwise the images of the terms are congruent or bisimilar. This particular observation is relevant for the proof of Theorem 40; remark that it does not hold for the proof of Theorem 28, where $\mathrel{\underset{\sim}{\sqsupset}}_{\pi}$ is also used for $!P \mathrel{\underset{\sim}{\sqsupset}}_{\pi} P$.

Example 36 Simulating the reduction of Example 10, using the semantic translation, runs as in Figure 3. As suggested by the proof of Theorem 35, the $\mathrel{\underset{\sim}{\sqsupset}}_{\pi}$ steps correspond to $(\mathit{cap}\dagger)$ and $(\dagger\mathit{cap})$.

This observation leads to:

Theorem 37 (Operational completeness for $\llbracket\cdot\rrbracket_{\mathrm{S}}$ with respect to $\rightarrow_{\mathcal{LK}}$) *If $\llbracket P \rrbracket_{\mathrm{S}} \rightarrow_{\pi} Q$ then there exists $P' \in \mathcal{LK}$, R such that $Q \rightarrow_{\pi}^{*} R$, $!R \approx \llbracket P' \rrbracket_{\mathrm{S}}$, and $P \rightarrow_{\mathcal{LK}}^{*} P'$.*

Proof From Remark 33 we know that the encoding $\llbracket\cdot\rrbracket_{\mathrm{S}}$ generates a flat parallel composition of processes of the shape

$$
!x(w).\overline{a}\langle w\rangle \qquad !\overline{\alpha}\langle y,\gamma\rangle \qquad !x(v,d).(!\alpha\rightarrow v \mid !d\rightarrow y) \qquad !\alpha\rightarrow x
$$

where all channel names are coming from the interpreted $\mathcal{LK}$-terms, and synchronisation over these channel names is possible only if generated by the interpretation of *cut*s. By

$$
\begin{array}{lll}
\llbracket (\widehat{z}P\widehat{\delta}\cdot\gamma)\,\widehat{\gamma} \dagger \widehat{u}(Q\widehat{\tau}\,[u]\,\widehat{w}R) \rrbracket_{\mathrm{S}} & \triangleq & \\
(\nu\gamma u)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to u \mid \llbracket Q\widehat{\tau}\,[u]\,\widehat{w}R \rrbracket_{\mathrm{S}}) & \triangleq & \\
(\nu\gamma u)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to u \mid (\nu\tau w)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !u(v,d).(!\tau\to v \mid !d\to w) \mid \llbracket R \rrbracket_{\mathrm{S}})) & \approx & (34) \\
(\nu\gamma y)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to y \mid (\nu\tau w)\,((\nu\gamma u)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to u \mid \llbracket Q \rrbracket_{\mathrm{S}}) \mid & & \\
\qquad\qquad !y(v,d).(!\tau\to v \mid !d\to w) \mid (\nu\gamma u)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to u \mid \llbracket R \rrbracket_{\mathrm{S}}))) & \sqsupseteq_{\pi}, \triangleq & (=_{\alpha}) \\
(\nu\gamma y)\,(\llbracket \widehat{x}P\widehat{\rho}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to y \mid (\nu\tau w)\,((\nu\gamma u)\,((\nu z\delta)\,(\llbracket \langle x\cdot\rho\rangle \rrbracket_{\mathrm{S}} \mid !\overline{\gamma}\langle z,\delta\rangle) \mid !\gamma\to u \mid & & \\
\qquad\qquad !u(w).\overline{\tau}\langle w\rangle) \mid !y(v,d).(!\tau\to v \mid !d\to w) \mid \llbracket R \rrbracket_{\mathrm{S}})) & \approx & (\gamma,u) \\
(\nu\gamma y)\,(\llbracket \widehat{x}P\widehat{\rho}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\gamma\to y \mid (\nu\tau w)\,((\nu z\delta)\,(\llbracket \langle x\cdot\rho\rangle \rrbracket_{\mathrm{S}} \mid !\overline{\tau}\langle z,\delta\rangle) \mid & & \\
\qquad\qquad !y(v,d).(!\tau\to v \mid !d\to w) \mid \llbracket R \rrbracket_{\mathrm{S}})) & \triangleq & \\
(\nu\gamma y)\,((\nu x\rho)\,(\llbracket \langle x\cdot\rho\rangle \rrbracket_{\mathrm{S}} \mid !\overline{\gamma}\langle x,\rho\rangle) \mid !\gamma\to y \mid & & \\
\qquad\qquad (\nu\tau w)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\tau \rrbracket_{\mathrm{S}} \mid !y(v,d).(!\tau\to v \mid !d\to w) \mid \llbracket R \rrbracket_{\mathrm{S}})) & \approx & (\gamma,y) \\
(\nu\gamma y)\,((\nu x\rho)\,(\llbracket \langle x\cdot\rho\rangle \rrbracket_{\mathrm{S}} \mid (\nu\tau w)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\gamma \rrbracket_{\mathrm{S}} \mid !\tau\to x \mid !\rho\to w \mid !\llbracket R \rrbracket_{\mathrm{S}}))) & \equiv & \\
(\nu\tau x)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\tau \rrbracket_{\mathrm{S}} \mid !\tau\to x \mid (\nu\rho w)\,(!x(w).\overline{\rho}\langle w\rangle \mid !\rho\to w \mid \llbracket R \rrbracket_{\mathrm{S}})) & \triangleq & \\
\llbracket (\widehat{z}P\widehat{\delta}\cdot\tau)\,\widehat{\tau} \dagger \widehat{x}(\langle x\cdot\rho\rangle\,\widehat{\rho} \dagger \widehat{w}R) \rrbracket_{\mathrm{S}} & \approx & (34) \\
(\nu\rho w)\,((\nu\tau x)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\tau \rrbracket_{\mathrm{S}} \mid !\tau\to x \mid !x(w).\overline{\rho}\langle w\rangle) \mid !\rho\to w \mid & & \\
\qquad\qquad (\nu\tau z)\,(\llbracket \widehat{z}P\widehat{\delta}\cdot\tau \rrbracket_{\mathrm{S}} \mid !\tau\to z \mid \llbracket R \rrbracket_{\mathrm{S}})) & \sqsupseteq_{\pi}, \triangleq & \\
(\nu\rho w)\,((\nu\tau x)\,((\nu z\delta)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\overline{\tau}\langle z,\delta\rangle) \mid !\tau\to x \mid !x(w).\overline{\rho}\langle w\rangle) \mid !\rho\to w \mid \llbracket R \rrbracket_{\mathrm{S}}) & \approx & (\tau,x,\rho,w) \\
(\nu z\delta)\,(\llbracket P \rrbracket_{\mathrm{S}} \mid !\overline{\sigma}\langle z,\delta\rangle) & &
\end{array}
$$

Fig. 3 Running the semantic translation of the term of Example 10

Theorem 35, all synchronisations that become possible later correspond to interpreted *cut*s as well. □

As mentioned in Section 2, Gentzen has shown that the innermost reduction strategy on *cut*-elimination for LK is normalising; in the context of $\mathcal{LK}$, this corresponds to showing that the innermost reduction strategy $\to_{\mathrm{I}}$ on $\mathcal{LK}$ is normalising for *typeable* terms.

Proposition 38 (Gentzen's Hauptsatz Result for $\mathcal{LK}$) If P $:\cdot$ $\Gamma \vdash \Delta$ (so P is typeable), then there exists Q in normal form such that $P \to_{\mathrm{I}}^{*} Q$.

We will show the equivalent of this result in the setting of the semantic interpretation. We first show:

Lemma 39 If P is in normal form, then so is $\llbracket P \rrbracket_{\mathrm{S}}$.

Proof First, notice that in P, the alphabets of sockets and plugs are distinct, and so are the names of input and output channels in $\llbracket P \rrbracket_{\mathrm{S}}$. The only exceptions are when P is a cut $Q\widehat{\alpha} \dagger \widehat{z}R$ and $\llbracket Q\widehat{\alpha} \dagger \widehat{z}R \rrbracket_{\mathrm{S}} = (\nu\alpha z)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\alpha\to z \mid \llbracket R \rrbracket_{\mathrm{S}})$ where the forwarder $!\alpha\to z = \alpha(w).\overline{z}\langle w\rangle$ is added; notice that here the output α is used as input, and the input z as output. Also in the synchronisation cell inside the interpretation of the import $Q\widehat{\alpha}\,[x]\,\widehat{z}R$, $(\nu\alpha z)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !x(v,d).(!\alpha\to v \mid !d\to z) \mid \llbracket R \rrbracket_{\mathrm{S}})$, this reversal is there. However, since in both cases α and z are restricted, these are not 'reachable' from outside, so can be ignored in this reasoning.

We continue by induction on the structure of terms; notice that P cannot be a cut.

$P = \langle x\cdot\alpha\rangle$: Since $\llbracket \langle x\cdot\alpha\rangle \rrbracket_{\mathrm{S}} = !x(w).\overline{\alpha}\langle w\rangle$, this is immediate.

$P = \widehat{x}Q\widehat{\alpha}\cdot\beta$: $\llbracket \widehat{x}Q\widehat{\alpha}\cdot\beta \rrbracket_{\mathrm{S}} = (\nu x\alpha)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle)$, and by induction, $\llbracket Q \rrbracket_{\mathrm{S}}$ is in normal form. Since β is not used inside $\llbracket Q \rrbracket_{\mathrm{S}}$ as input, no synchronisation is possible in $(\nu x\alpha)\,(\llbracket Q \rrbracket_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle)$ over β and also $\llbracket \widehat{x}Q\widehat{\alpha}\cdot\beta \rrbracket_{\mathrm{S}}$ is in normal form.

$P = Q\widehat{\alpha}\,[x]\,\widehat{z}R$: $[\![Q\widehat{\alpha}\,[x]\,\widehat{z}R]\!]_{\mathrm{S}} = (\nu\alpha z)\,([\![Q]\!]_{\mathrm{S}} \mid !\,x(v,d).(!\alpha{\rightarrow}v \mid !d{\rightarrow}z) \mid [\![R]\!]_{\mathrm{S}})$, and by induction, $[\![Q]\!]_{\mathrm{S}}$ and $[\![R]\!]_{\mathrm{S}}$ are in normal form. Since x is not an output in either $[\![Q]\!]_{\mathrm{S}}$ or $[\![R]\!]_{\mathrm{S}}$, and input and output names are distinct, no synchronisation is possible between $[\![Q]\!]_{\mathrm{S}}$ and $[\![R]\!]_{\mathrm{S}}$, so also $[\![Q\widehat{\alpha}\,[x]\,\widehat{z}R]\!]_{\mathrm{S}}$ is in normal form. □

Now for termination, we can show:

Theorem 40 *If $P \rightarrow_{\mathrm{I}}^{*} Q$, and Q is in normal from, then there exists an R in normal form such that $[\![P]\!]_{\mathrm{S}} \approx \mathsf{R}$ and $\mathsf{R} \sqsupseteq_{\pi} [\![Q]\!]_{\mathrm{S}}$.*

Proof By Lemma 39, $[\![Q]\!]_{\mathrm{S}}$ is in normal form. By Theorem 35, there exists R such that $[\![P]\!]_{\mathrm{S}} \approx \mathsf{R}$ and $\mathsf{R} \sqsupseteq_{\pi} [\![Q]\!]_{\mathrm{S}}$. By the proof of that theorem, $\sqsupseteq_{\pi}$ is only needed in part $(cap\dagger)$ and $(\dagger cap)$, where we eliminate part of a process, otherwise the images of the terms are congruent or bisimilar. Since now only innermost reduction steps in $P \rightarrow_{\mathrm{I}}^{*} Q$ are simulated, $\approx$ deals with communications between processes in normal form, so by Lemma 39, whenever $(cap\dagger)$ or $(\dagger cap)$ are simulated, the 'discarded' term corresponds to a process in normal form. So when reaching $[\![Q]\!]_{\mathrm{S}}$, the whole process R is in normal form. □

Combining this with Gentzen's Hauptsatz result, we get:

Theorem 41 (Preservation of typeable termination) *Let P be a typeable term, then there exists R in normal form such that $[\![P]\!]_{\mathrm{S}} \approx \mathsf{R}$.*

Proof If P is typeable, then by Gentzen's result, innermost reduction terminates, so there exists Q in normal form such that $P \rightarrow_{\mathrm{I}}^{*} Q$; then, by Theorem 40, there exists an R in normal form such that $[\![P]\!]_{\mathrm{S}} \approx \mathsf{R}$. □

Notice that we cannot show this result for the natural translation this way, since by Example 31 not every term in $\rightarrow_{\mathrm{H}}$ normal form is interpreted by a process in normal form.

6 Type assignment

In this section, we discuss a notion of type assignment $\vdash_{\pi}$ for processes in $\pi_{\langle\rangle}$, as first presented in [7], that describes the '*input-output interface*' of a process. Typeability of a process is expressed via the ternary relation

$$\mathsf{P} : \Gamma \vdash_{\pi} \Delta$$

(so almost identical to that for $\mathcal{LK}$), where P is a process that is said to be the *witness* to the judgement $\Gamma \vdash \Delta$, the left context Γ contains pairs of channel names and types for all the input channels of P, and the right context Δ for its output channels; since in P a channel can be used for both, it can appear in both contexts. Our system thereby gives an abstract functional translation of processes by stating the connectability of a process via giving the names of the available (connectable) channels and their types.

We will show that, if P is a witness to a judgement (in $\vdash_{\mathrm{LK}}$), then its translations via $[\![\cdot]\!]_{\mathrm{N}}$ and $[\![\cdot]\!]_{\mathrm{S}}$ are as well. Together with the preservation results we have shown above, this implies that we can not only interpret reduction in LK through synchronisation (similarly to what was done, for example, in [41] for the lazy λ-calculus), but actually show that the processes we create through our interpretations accurately represent the *actual proofs*, so our synchronisations correctly model *cut*-elimination, and transforms a proof into a proof.

Definition 42 (Implicative type assignment for $\pi_{\langle\rangle}$ [7])

1. The types and contexts we consider for $\pi_{\langle\rangle}$ are defined like those of Definition 3, generalised to names, but allowing both Roman and Greek names on *both* sides of the turnstyle.
2. Type assignment for π-calculus is defined by the following sequent system:

$$(\mathsf{0}) : \frac{}{\mathsf{0} : \ \vdash} \qquad (!) : \frac{P : \Gamma \vdash \Delta}{!P : \Gamma \vdash \Delta} \qquad (out) : \frac{}{\overline{a}\langle b\rangle : b{:}A \vdash a{:}A, b{:}A}\ (a \neq b)$$

$$(\nu) : \frac{P : \Gamma, a{:}A \vdash a{:}A, \Delta}{(\nu a)\, P : \Gamma \vdash \Delta} \qquad (\mathit{pair\text{-}out}) : \frac{}{\overline{a}\langle b,c\rangle : b{:}A \vdash a{:}A{\to}B, c{:}B}\ (b \neq a, c)$$

$$(|) : \frac{P_1 : \Gamma \vdash \Delta \ \cdots \ P_n : \Gamma \vdash \Delta}{P_1 \mid \cdots \mid P_n : \Gamma \vdash \Delta} \qquad (in) : \frac{P : \Gamma, x{:}A \vdash x{:}A, \Delta}{a(x).P : \Gamma, a{:}A \vdash \Delta}$$

$$(W) : \frac{P : \Gamma \vdash \Delta}{P : \Gamma' \vdash \Delta'}\ (\Gamma' \supseteq \Gamma, \Delta' \supseteq \Delta) \qquad (\mathit{let}) : \frac{P : \Gamma, y{:}B \vdash x{:}A, \Delta}{\mathit{let}\,\langle x,y\rangle{=}z\ \mathit{in}\ P : \Gamma, z{:}A{\to}B \vdash \Delta}\ (y,z \notin \Delta; x \notin \Gamma)$$

3. As usual, we write $P : \Gamma \vdash_{\pi} \Delta$ if there exists a derivation using these rules that has the expression $P : \Gamma \vdash \Delta$ in the conclusion.

This notion is novel in that it assigns to channels the type of the input or output that is sent over the channel; in that it differs from normal notions, that would state:

$$\frac{}{\overline{a}\langle b\rangle : \Gamma, b{:}A \vdash a{:}\mathsf{ch}(A), \Delta} \qquad \text{or} \qquad \frac{}{\overline{a}\langle b\rangle : \Gamma, b{:}A \vdash_{\pi} a{:}[A], \Delta}$$

In order to be able to interpret LK, types in our system need not be decorated with channel information, but will express functionality instead.

This notion is a true type assignment system which does not (directly) relate back to LK.[12] For example, rule $(\mathsf{0})$ makes $\mathsf{0}$ a witness to an unprovable judgement, and rules $(|)$ and $(!)$ do not change the contexts, so do not correspond to any rule in the logic, not even to a $\lambda\mu$-style [42] activation step. Moreover, rule (ν) just removes a formula, and rule $(\mathit{pair\text{-}out})$ is clearly not an instance of an axiom in LK; notice that that rule does not directly correspond to the logical rule $({\Rightarrow}R)$, as that $(\mathit{pair\text{-}in})$ does not directly correspond to $({\Rightarrow}L)$. However, in view of the intended property - preservation of context assignment - this is not problematic, since we will not map rules to rules, but proofs to type derivations. This apparent discrepancy is solved by Theorem 44.

The presence of *weakening* allows us to be a little less precise when we construct derivations, and allow for rules to join contexts, by using, for example, the rule

$$(|) : \frac{P : \Gamma_1 \vdash_{\pi} \Delta_1 \quad Q : \Gamma_2 \vdash_{\pi} \Delta_2}{P \mid Q : \Gamma_1, \Gamma_2 \vdash_{\pi} \Delta_1, \Delta_2}$$

so switching, without any scruples, to multiplicative style, whenever convenient.

Notice that rule (let) is formulated with a single name z; as we pointed out above, the process $\mathit{let}\,\langle x,y\rangle{=}\langle a,b\rangle\ \mathit{in}\ P$ is congruent to $P[a/x, b/y]$, and since we consider processes modulo congruence, we do not need a rule that types the former. Notice that the '*input-output interface of a π-process*' property is nicely preserved by all the rules; it also explains how the handling of pairs is restricted by the type system to the rules (let) and $(\mathit{pair\text{-}out})$.

[12] We leave the exploration of the logical contents of this system for future work.

Example 43 We can derive

$$\dfrac{\dfrac{\begin{array}{c}\bigtriangledown\\ P : \Gamma, y{:}B \vdash_\pi x{:}A, \Delta\end{array}}{\mathit{let}\,\langle x,y\rangle = z\ \mathit{in}\ P : \Gamma, z{:}A\to B \vdash_\pi \Delta}\ (\mathit{let})}{a(z).\mathit{let}\,\langle x,y\rangle = z\ \mathit{in}\ P : \Gamma, a{:}A\to B \vdash_\pi \Delta}\ (\mathit{in})$$

so the following rule is derivable:

$$(\mathit{pair\text{-}in}) : \ \dfrac{P : \Gamma, y{:}B \vdash x{:}A, \Delta}{a(x,y).P : \Gamma, a{:}A\to B \vdash \Delta}\ (y,a \notin \Delta,\ x \notin \Gamma)$$

We now come to the main soundness result for our notion of type assignment for $\pi_{\langle\rangle}$. Since in the synchronisation step $\overline{a}\langle b,c\rangle \mid a(x,y).Q \to_\pi Q[b/x, c/y]$, in $\overline{a}\langle b,c\rangle$ the name b is used for input, and c for output whereas their role is reversed in $Q[b/x, c/y]$, we cannot show a straightforward witness reduction result.[13]

The following theorem shows that the natural translation $\llbracket\cdot\rrbracket_{\mathrm{N}}$ preserves assignable types.

Theorem 44 (Type preservation for the natural interpretation) *If* $P \mathrel{:\!\cdot} \Gamma \vdash \Delta$, *then* $\llbracket P\rrbracket_{\mathrm{N}} : \Gamma \vdash_\pi \Delta$.

Proof By induction on the structure of terms in $\mathcal{LK}$.

$\langle x{\cdot}\alpha\rangle$: Then $\llbracket\langle x{\cdot}\alpha\rangle\rrbracket_{\mathrm{N}} = x(w).\overline{\alpha}\langle w\rangle$, and the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{}{\langle x{\cdot}\alpha\rangle \mathrel{:\!\cdot} \Gamma, x{:}A \vdash \alpha{:}A, \Delta}\ (\mathit{cap})$$

Notice that

$$\dfrac{\dfrac{\dfrac{}{\overline{\alpha}\langle w\rangle : w{:}A \vdash_\pi \alpha{:}A, w{:}A}\ (\mathit{out})}{x(w).\overline{\alpha}\langle w\rangle : x{:}A \vdash_\pi \alpha{:}A}\ (\mathit{in})}{x(w).\overline{\alpha}\langle w\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}A, \Delta}\ (W)$$

and $\llbracket\langle x{\cdot}\alpha\rangle\rrbracket_{\mathrm{N}} = x(w).\overline{\alpha}\langle w\rangle$.

$\widehat{x}P\widehat{\alpha}{\cdot}\beta$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{\begin{array}{c}\bigtriangledown\\ P \mathrel{:\!\cdot} \Gamma, x{:}A \vdash \alpha{:}B, \Delta\end{array}}{\widehat{x}P\widehat{\alpha}{\cdot}\beta \mathrel{:\!\cdot} \Gamma \vdash \beta{:}A\to B, \Delta}\ (\mathit{exp})$$

Then, by induction, $\llbracket P\rrbracket_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \Delta$, and we can construct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\bigtriangledown\\ \llbracket P\rrbracket_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \Delta\end{array}}{!\,\llbracket P\rrbracket_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \Delta}\ (!) \qquad \dfrac{}{\overline{\beta}\langle x,\alpha\rangle : x{:}A \vdash_\pi \alpha{:}B, \beta{:}A\to B}\ (\mathit{pair\text{-}out})}{!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\beta}\langle x,\alpha\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \beta{:}A\to B, \Delta}\ (\mid)}{(\nu\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\beta}\langle x,\alpha\rangle) : \Gamma, x{:}A \vdash_\pi \beta{:}A\to B, \Delta}\ (\nu)}{(\nu x\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\beta}\langle x,\alpha\rangle) : \Gamma \vdash_\pi \beta{:}A\to B, \Delta}\ (\nu)$$

and $\llbracket\widehat{x}P\widehat{\alpha}{\cdot}\beta\rrbracket_{\mathrm{N}} = (\nu x\alpha)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\beta}\langle x,\alpha\rangle)$.

[13] We could show it for a notion of type assignment that no longer separates inputs from outputs (so with judgements like $\Gamma, \Delta \vdash P$ rather than $P : \Gamma \vdash_\pi \Delta$), but consider this out of scope for this paper.

$P\widehat{\alpha}\,[y]\,\widehat{x}Q$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{P :\cdot\ \Gamma \vdash \alpha{:}A,\Delta \qquad Q :\cdot\ \Gamma, x{:}B \vdash \Delta}{P\widehat{\alpha}\,[y]\,\widehat{x}Q :\cdot\ \Gamma, y{:}A{\to}B \vdash \Delta}\,(imp)$$

Then, by induction, we have derivations for $[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A,\Delta$ and $[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}B \vdash_\pi \Delta$, and we can construct:

$$\dfrac{\dfrac{\dfrac{[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A,\Delta}{!\,[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A,\Delta}\,(!) \qquad \dfrac{[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}B \vdash_\pi \Delta}{!\,[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}B \vdash_\pi \Delta}\,(!)}{!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}B \vdash_\pi \alpha{:}A,\Delta}\,(\mid)}{y(\alpha,x).(!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}}) : \Gamma, y{:}A{\to}B \vdash_\pi \Delta}\,(in)$$

and $[\![P\widehat{\alpha}\,[y]\,\widehat{x}Q]\!]_{\mathrm{N}} = y(\alpha,x).(!\,[\![P]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}})$.

$P\widehat{\alpha} \dagger \widehat{x}Q$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{P :\cdot\ \Gamma \vdash \alpha{:}A,\Delta \qquad Q :\cdot\ \Gamma, x{:}A \vdash \Delta}{P\widehat{\alpha} \dagger \widehat{x}Q :\cdot\ \Gamma \vdash \Delta}\,(cut)$$

By induction, we have derivations for both $[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A,\Delta$ - and since x does not occur in P, also for $[\![P[x/\alpha]]\!]_{\mathrm{N}} : \Gamma \vdash_\pi x{:}A,\Delta$ - and $[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta$. Then we can construct:

$$\dfrac{\dfrac{\dfrac{[\![P[x/\alpha]]\!]_{\mathrm{N}} : \Gamma \vdash_\pi x{:}A,\Delta}{!\,[\![P[x/\alpha]]\!]_{\mathrm{N}} : \Gamma \vdash_\pi x{:}A,\Delta}\,(!) \qquad \dfrac{[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta}{!\,[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta}\,(!)}{!\,[\![P[x/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi x{:}A,\Delta}\,(\mid)}{(\nu x)\,(!\,[\![P[x/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}}) : \Gamma \vdash_\pi \Delta}\,(\nu)$$

and $[\![P\widehat{\alpha} \dagger \widehat{x}Q]\!]_{\mathrm{N}} = (\nu x)\,(!\,[\![P[x/\alpha]]\!]_{\mathrm{N}} \mid !\,[\![Q]\!]_{\mathrm{N}})$. □

We can also show that the semantic translation $[\![\cdot]\!]_{\mathrm{S}}$ preserves assignable types.

Theorem 45 (Type preservation for the semantic interpretation) *If* $P :\cdot\ \Gamma \vdash \Delta$*, then* $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \Delta$.

Proof By induction on the structure of terms in $\mathcal{LK}$.

$\langle x{\cdot}\alpha\rangle$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{}{\langle x{\cdot}\alpha\rangle :\cdot\ \Gamma, x{:}A \vdash \alpha{:}A,\Delta}\,(Ax)$$

Notice that

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{\alpha}\langle w\rangle : w{:}A \vdash_\pi \alpha{:}A, w{:}A}\,(out)}{x(w).\overline{\alpha}\langle w\rangle : x{:}A \vdash_\pi \alpha{:}A}\,(in)}{!\,x(w).\overline{\alpha}\langle w\rangle : x{:}A \vdash_\pi \alpha{:}A}\,(!)}{!\,x(w).\overline{\alpha}\langle w\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}A,\Delta}\,(W)$$

and $[\![\langle x{\cdot}\alpha\rangle]\!]_{\mathrm{S}} = !\,x(w).\overline{\alpha}\langle w\rangle$.

$\widehat{x}P\widehat{\alpha}\cdot\beta$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{\begin{array}{c}\triangledown\\ P \mathrel{:\!\cdot} \Gamma, x{:}A \vdash \alpha{:}B, \Delta\end{array}}{\widehat{x}P\widehat{\alpha}\cdot\beta \mathrel{:\!\cdot} \Gamma \vdash \beta{:}A\to B, \Delta}\,(\mathit{exp})$$

Then, by induction, $[\![P]\!]_{\mathrm{S}} : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \Delta$, and we can construct:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\triangledown\\ [\![P]\!]_{\mathrm{S}} : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \Delta\end{array} \qquad \dfrac{\dfrac{}{\overline{\beta}\langle x,\alpha\rangle : x{:}A \vdash_\pi \alpha{:}B, \beta{:}A\to B}\,(\mathit{pair\text{-}out})}{!\overline{\beta}\langle x,\alpha\rangle : x{:}A \vdash_\pi \alpha{:}B, \beta{:}A\to B}\,(!)}{[\![P]\!]_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}B, \beta{:}A\to B, \Delta}\,(\mid)}{(\nu\alpha)\,([\![P]\!]_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) : \Gamma, x{:}A \vdash_\pi \beta{:}A\to B, \Delta}\,(\nu)}{(\nu x\alpha)\,([\![P]\!]_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle) : \Gamma \vdash_\pi \beta{:}A\to B, \Delta}\,(\nu)$$

and $[\![\widehat{x}P\widehat{\alpha}\cdot\beta]\!]_{\mathrm{S}} = (\nu x\alpha)\,([\![P]\!]_{\mathrm{S}} \mid !\overline{\beta}\langle x,\alpha\rangle)$

$P\widehat{\alpha}\,[y]\,\widehat{x}Q$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{\begin{array}{c}\triangledown\\ P \mathrel{:\!\cdot} \Gamma \vdash \alpha{:}A, \Delta\end{array} \qquad \begin{array}{c}\triangledown\\ Q \mathrel{:\!\cdot} \Gamma, x{:}B \vdash \Delta\end{array}}{P\widehat{\alpha}\,[y]\,\widehat{x}Q \mathrel{:\!\cdot} \Gamma, y{:}A\to B \vdash \Delta}\,(\mathit{imp})$$

Then, by induction, we have derivations for $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A, \Delta$ and $[\![Q]\!]_{\mathrm{S}} : \Gamma, x{:}B \vdash_\pi \Delta$, and we can construct:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\triangledown\\ [\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A, \Delta\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{v}\langle w\rangle : w{:}A \vdash_\pi v{:}A, w{:}A}\,(\mathit{out})}{\alpha \mathbin{\cdot\!\!\!\to} v : \alpha{:}A \vdash_\pi v{:}A}\,(\mathit{in})}{!\alpha \mathbin{\cdot\!\!\!\to} v : \alpha{:}A \vdash_\pi v{:}A}\,(!) \quad \dfrac{\dfrac{\dfrac{}{\overline{x}\langle w\rangle : w{:}B \vdash_\pi x{:}B, w{:}B}\,(\mathit{out})}{d \mathbin{\cdot\!\!\!\to} x : d{:}B \vdash_\pi x{:}B}\,(\mathit{in})}{!d \mathbin{\cdot\!\!\!\to} x : d{:}B \vdash_\pi x{:}B}\,(!)}{!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x : \alpha{:}A, d{:}B \vdash_\pi x{:}B, v{:}A}\,(\mid)}{y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) : \alpha{:}A, y{:}A\to B \vdash_\pi x{:}B}\,(\mathit{in})}{!y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) : \alpha{:}A, y{:}A\to B \vdash_\pi x{:}B}\,(!) \quad \begin{array}{c}\triangledown\\ [\![Q]\!]_{\mathrm{S}} : \Gamma, x{:}B \vdash_\pi \Delta\end{array}}{[\![P]\!]_{\mathrm{S}} \mid !y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) \mid [\![Q]\!]_{\mathrm{S}} : \Gamma, x{:}B, \alpha{:}A, y{:}A\to B \vdash_\pi x{:}B, \alpha{:}A, \Delta}\,(\mid)}{(\nu x)\,([\![P]\!]_{\mathrm{S}} \mid !y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) \mid [\![Q]\!]_{\mathrm{S}}) : \Gamma, \alpha{:}A, y{:}A\to B \vdash_\pi \alpha{:}A, \Delta}\,(\nu)}{(\nu\alpha x)\,([\![P]\!]_{\mathrm{S}} \mid !y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) \mid [\![Q]\!]_{\mathrm{S}}) : \Gamma, y{:}A\to B \vdash_\pi \Delta}\,(\nu)$$

Notice that $[\![P\widehat{\alpha}\,[y]\,\widehat{x}Q]\!]_{\mathrm{S}} = (\nu\alpha x)\,([\![P]\!]_{\mathrm{S}} \mid !y(v,d).(!\alpha \mathbin{\cdot\!\!\!\to} v \mid !d \mathbin{\cdot\!\!\!\to} x) \mid [\![Q]\!]_{\mathrm{S}})$.

$P\widehat{\alpha} \dagger \widehat{x}Q$: Then the $\mathcal{LK}$-derivation is shaped like:

$$\dfrac{\begin{array}{c}\triangledown\\ P \mathrel{:\!\cdot} \Gamma \vdash \alpha{:}A, \Delta\end{array} \qquad \begin{array}{c}\triangledown\\ Q \mathrel{:\!\cdot} \Gamma, x{:}A \vdash \Delta\end{array}}{P\widehat{\alpha} \dagger \widehat{x}Q \mathrel{:\!\cdot} \Gamma \vdash \Delta}\,(\mathit{cut})$$

By induction, we have derivations for both $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A,\Delta$ and $[\![Q]\!]_{\mathrm{S}} : \Gamma, x{:}A \vdash_\pi \Delta$. Then we can construct:

$$\dfrac{\dfrac{\dfrac{}{[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A,\Delta} \qquad \dfrac{\dfrac{\dfrac{}{\overline{x}\langle w\rangle : w{:}A \vdash_\pi x{:}A, w{:}A}\;(out)}{\alpha \rightarrow x : \alpha{:}A \vdash_\pi x{:}A}\;(in)}{!\alpha \rightarrow x : \alpha{:}A \vdash_\pi x{:}A}\;(!) \qquad \dfrac{}{[\![Q]\!]_{\mathrm{S}} : \Gamma, x{:}A \vdash_\pi \Delta}}{\dfrac{[\![P]\!]_{\mathrm{S}} \mid !\alpha \rightarrow x \mid [\![Q]\!]_{\mathrm{S}} : \Gamma, \alpha{:}A, x{:}A \vdash_\pi \alpha{:}A, x{:}A, \Delta}{\dfrac{(\nu x)\,([\![P]\!]_{\mathrm{S}} \mid !\alpha \rightarrow x \mid [\![Q]\!]_{\mathrm{S}}) : \Gamma, \alpha{:}A \vdash_\pi \alpha{:}A, \Delta}{(\nu \alpha x)\,([\![P]\!]_{\mathrm{S}} \mid !\alpha \rightarrow x \mid [\![Q]\!]_{\mathrm{S}}) : \Gamma \vdash_\pi \Delta}\;(\nu)}\;(\nu)}\;(\mid)}{}$$

and $[\![P\widehat{\alpha} \dagger \widehat{x}Q]\!]_{\mathrm{S}} = (\nu \alpha x)\,([\![P]\!]_{\mathrm{S}} \mid !\alpha \rightarrow x \mid [\![Q]\!]_{\mathrm{S}})$. □

The following is a direct result of the last two theorems:

Proposition 46 (Simulation of cut-elimination) Assume $P \rightarrow_{\mathcal{LK}} Q$, then :

1. *if $[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \Delta$, then $[\![Q]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \Delta$.*
2. *if $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \Delta$, then $[\![Q]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \Delta$.*

Proof By Theorem 11, 44, and 45. □

We will show that our notion of type assignment is preserved for reductions in the image of our translations, for which we first show a contraction result.

Lemma 47 (Contraction)

1. *If a does not occur in Q, $a \neq b$, and $(\nu b)\,\overline{a}\langle b\rangle \mid a(x).Q : \Gamma, a{:}C \vdash_\pi a{:}C, \Delta$, then $(\nu b)\,(Q[b/x]) : \Gamma \vdash_\pi \Delta$.*
2. *If a does not occur in P, $a \neq e$, and $(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(x).\overline{e}\langle x\rangle : \Gamma, a{:}C \vdash_\pi a{:}C, \Delta$, then $(\nu bc)\,(P \mid \overline{e}\langle b,c\rangle) : \Gamma \vdash_\pi \Delta$.*
3. *If a does not occur in P and Q and $(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(x,y).Q : \Gamma, a{:}C \vdash_\pi a{:}C, \Delta$, then $(\nu bc)\,(P \mid Q[b/x, c/y]) : \Gamma \vdash_\pi \Delta$.*
4. *If a does not occur in P and $(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(v,d).(!x \rightarrow v \mid !d \rightarrow y) : \Gamma, a{:}C \vdash_\pi a{:}C, \Delta$, then $(\nu bc)\,(P \mid !x \rightarrow b \mid !c \rightarrow y) : \Gamma \vdash_\pi \Delta$.*

Proof 1. Then the derivation is shaped like:

$$\dfrac{\dfrac{\dfrac{}{\overline{a}\langle b\rangle : b{:}A \vdash_\pi a{:}A, b{:}A}\;(out)}{(\nu b)\,\overline{a}\langle b\rangle : \ \vdash_\pi a{:}A}\;(\nu) \qquad \dfrac{\dfrac{}{Q : \Gamma', x{:}A \vdash_\pi x{:}A, \Delta'}}{a(x).Q : \Gamma', a{:}A \vdash_\pi \Delta'}\;(in)}{(\nu b)\,\overline{a}\langle b\rangle \mid a(x).Q : \Gamma', a{:}A \vdash_\pi a{:}A, \Delta'}\;(\mid)$$

Since b is restricted, it does not occur in Q, also $Q[b/x] : \Gamma, b{:}A \vdash_\pi b{:}A, \Delta$, and we can construct:

$$\dfrac{\dfrac{}{Q[b/x] : \Gamma', b{:}A \vdash_\pi b{:}A, \Delta'}}{(\nu b)\,Q[b/x] : \Gamma' \vdash_\pi \Delta'}\;(\nu)$$

2. Then the derivation is shaped like:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\boxed{\qquad}\\ P : \Gamma' \vdash_\pi \Delta'\end{array} \qquad \dfrac{}{\overline{a}\langle b,c\rangle : b{:}A \vdash_\pi a{:}A{\to}B, c{:}B}\,(\textit{pair-out})}{P \mid \overline{a}\langle b,c\rangle : \Gamma', b{:}A \vdash_\pi a{:}A{\to}B, c{:}B, \Delta'}\,(\mid)}{(\nu c)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma', b{:}A \vdash_\pi a{:}A{\to}B, \Delta'}\,(\nu)}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma' \vdash_\pi a{:}A{\to}B, \Delta'}\,(\nu) \qquad \dfrac{\dfrac{}{\overline{e}\langle x\rangle : x{:}A{\to}B \vdash_\pi e{:}A{\to}B, x{:}A{\to}B}\,(\textit{out})}{a(x).\overline{e}\langle x\rangle : a{:}A{\to}B \vdash_\pi e{:}A{\to}B}\,(\textit{in})}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(x).\overline{e}\langle x\rangle : \Gamma', a{:}A{\to}B \vdash_\pi a{:}A{\to}B, e{:}A{\to}B, \Delta'}\,(\mid)$$

Since b and c are restricted, they are different from e and we can construct:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\boxed{\qquad}\\ P : \Gamma' \vdash_\pi \Delta'\end{array} \qquad \dfrac{}{\overline{e}\langle b,c\rangle : b{:}A \vdash_\pi e{:}A{\to}B, c{:}B}\,(\textit{pair-out})}{P \mid \overline{e}\langle b,c\rangle : \Gamma', b{:}A \vdash_\pi e{:}A{\to}B, c{:}B, \Delta'}\,(\mid)}{(\nu c)\,(P \mid \overline{e}\langle b,c\rangle) : \Gamma', b{:}A \vdash_\pi e{:}A{\to}B, \Delta'}\,(\nu)}{(\nu bc)\,(P \mid \overline{e}\langle b,c\rangle) : \Gamma' \vdash_\pi e{:}A{\to}B, \Delta'}\,(\nu)$$

3. Since $a(x,y).Q$ is short for $a(z).\textit{let}\,\langle x,y\rangle{=}z \textit{ in } Q$, we have in fact a derivation of the shape

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\boxed{\qquad}\\ P : \Gamma \vdash_\pi \Delta\end{array} \qquad \dfrac{}{\overline{a}\langle b,c\rangle : \Gamma, b{:}A \vdash_\pi a{:}A{\to}B, c{:}B, \Delta}\,(\textit{pair-out})}{P \mid \overline{a}\langle b,c\rangle : \Gamma, b{:}A \vdash_\pi a{:}A{\to}B, \Delta}\,(\mid)}{(\nu c)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma, b{:}A \vdash_\pi a{:}A{\to}B, \Delta}\,(\nu)}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma \vdash_\pi a{:}A{\to}B, \Delta}\,(\nu) \qquad \dfrac{\dfrac{\begin{array}{c}\boxed{\qquad\qquad}\\ Q : \Gamma, y{:}B \vdash_\pi x{:}A, \Delta\end{array}}{\textit{let}\,\langle x,y\rangle{=}z \textit{ in } Q : \Gamma, z{:}A{\to}B \vdash_\pi \Delta}\,(\textit{let})}{a(z).\textit{let}\,\langle x,y\rangle{=}z \textit{ in } Q : \Gamma, a{:}A{\to}B \vdash_\pi \Delta}\,(\textit{in})}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(z).\textit{let}\,\langle x,y\rangle{=}z \textit{ in } Q : \Gamma, a{:}A{\to}B \vdash_\pi a{:}A{\to}B, \Delta}\,(\mid)$$

Since b and c are restricted, they do not occur in Q, so also $Q[b/x, c/y] : \Gamma, c{:}B \vdash_\pi b{:}A, \Delta$ and we can construct:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\boxed{\qquad}\\ P : \Gamma \vdash_\pi \Delta\end{array} \qquad \begin{array}{c}\boxed{\qquad\qquad}\\ Q[b/x, c/y] : \Gamma, c{:}B \vdash_\pi b{:}A, \Delta\end{array}}{P \mid Q[b/x, c/y] : \Gamma \vdash_\pi b{:}A, \Delta}\,(\mid)}{(\nu c)\,(P \mid Q[b/x, c/y]) : \Gamma \vdash_\pi b{:}A, \Delta}\,(\nu)}{(\nu bc)\,(P \mid Q[b/x, c/y]) : \Gamma \vdash_\pi \Delta}\,(\nu)$$

4. Then the derivation is of the shape

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\boxed{\qquad}\\ P : \Gamma' \vdash_\pi \Delta'\end{array} \qquad \dfrac{}{\overline{a}\langle b,c\rangle : b{:}A \vdash_\pi a{:}A{\to}B, c{:}B}\,(\textit{pair-out})}{P \mid \overline{a}\langle b,c\rangle : \Gamma', b{:}A \vdash_\pi a{:}A{\to}B, \Delta'}\,(\mid)}{(\nu c)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma', b{:}A \vdash_\pi a{:}A{\to}B, \Delta'}\,(\nu)}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) : \Gamma' \vdash_\pi a{:}A{\to}B, \Delta'}\,(\nu) \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{}{\overline{v}\langle w\rangle : w{:}A \vdash_\pi v{:}A, w{:}A}\,(\textit{out})}{x{\rightarrow}v : x{:}A \vdash_\pi v{:}A}\,(\textit{in})}{!\,x{\rightarrow}v : x{:}A \vdash_\pi v{:}A}\,(!)\\ \vdots\end{array} \qquad \dfrac{\dfrac{\dfrac{}{\overline{y}\langle w\rangle : w{:}B \vdash_\pi y{:}B, w{:}B}\,(\textit{out})}{d{\rightarrow}y : d{:}B \vdash_\pi y{:}B}\,(\textit{in})}{!\,d{\rightarrow}y : d{:}B \vdash_\pi y{:}B}\,(!)}{!\,x{\rightarrow}v \mid !\,d{\rightarrow}y : x{:}A, d{:}B \vdash_\pi y{:}B, v{:}A}\,(\mid)}{a(v,d).(!\,x{\rightarrow}v \mid !\,d{\rightarrow}y) : x{:}A, a{:}A{\to}B \vdash_\pi y{:}B}\,(\textit{in})}{(\nu bc)\,(P \mid \overline{a}\langle b,c\rangle) \mid a(v,d).(!\,x{\rightarrow}v \mid !\,d{\rightarrow}y) : \Gamma', x{:}A, a{:}A{\to}B \vdash_\pi a{:}A{\to}B, y{:}B, \Delta'}\,(\mid)$$

Since b and c are restricted, they are different from x and y and we can construct:

$$\dfrac{\dfrac{\dfrac{\overset{\bigtriangledown}{P : \Gamma' \vdash_\pi \Delta'} \qquad \dfrac{\dfrac{\dfrac{}{\overline{b}\langle w\rangle : w{:}A \vdash_\pi b{:}A, w{:}A}\,(\mathit{out})}{x \rightarrow b : x{:}A \vdash_\pi b{:}A}\,(\mathit{in})}{!x \rightarrow b : x{:}A \vdash_\pi b{:}A}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\overline{y}\langle w\rangle : w{:}B \vdash_\pi y{:}B, w{:}B}\,(\mathit{out})}{c \rightarrow y : c{:}B \vdash_\pi y{:}B}\,(\mathit{in})}{!c \rightarrow y : c{:}B \vdash_\pi y{:}B}\,(!)}{P \mid !x \rightarrow b \mid !c \rightarrow y : \Gamma', x{:}A, c{:}B \vdash_\pi b{:}A, y{:}B, \Delta'}\,(\mid)}{(\nu c)\,(P \mid !x \rightarrow b \mid !c \rightarrow y) : \Gamma', x{:}A \vdash_\pi b{:}A, y{:}B, \Delta'}\,(\nu)}{(\nu bc)\,(P \mid !x \rightarrow b \mid !c \rightarrow y) : \Gamma', x{:}A \vdash_\pi y{:}B, \Delta'}\,(\nu) \qquad \square$$

Using this result, we can also show a witness reduction result:

Theorem 48 *1. If $\llbracket P \rrbracket_{\mathrm{N}} : \Gamma \vdash_\pi \Delta$, and $\llbracket P \rrbracket_{\mathrm{N}} \rightarrow_\pi^* Q$, then $Q : \Gamma \vdash_\pi \Delta$.*
2. If $\llbracket P \rrbracket_{\mathrm{S}} : \Gamma \vdash_\pi \Delta$, and $\llbracket P \rrbracket_{\mathrm{S}} \rightarrow_\pi^* Q$, then $Q : \Gamma \vdash_\pi \Delta$.

Proof By Remark 24 and 33, Theorem 44 and 45, and Lemma 47. □

7 Expressing Negation

In this section we will look at the logical connective $\neg$, how it is dealt with within $\mathcal{LK}$,[14] and how to interpret it in the π-calculus. Together with the treatment of implication it is then possible to also express all other first-order logical connectives, but we will not deal with those explicitly here.

Definition 49 The sequent rules that correspond to negation are as follows:

$$(\neg R) : \dfrac{\Gamma, x{:}A \vdash \Delta}{\Gamma \vdash \alpha{:}\neg A, \Delta} \qquad\qquad (\neg L) : \dfrac{\Gamma \vdash \alpha{:}A, \Delta}{\Gamma, x{:}\neg A \vdash \Delta}$$

To extend the Curry-Howard isomorphism of $\mathcal{LK}$ also to negation, we follow the same approach as used for the arrow: a disappearing formula in a context corresponds to a connector that gets bound, and a formula that appears in a context corresponds to a connector that is introduced.

Definition 50 1. We extend $\mathcal{LK}$'s syntax with the following constructs:

$$\begin{array}{rcll} P & ::= & \ldots \mid \; x \cdot P\widehat{\alpha} & \textit{left inversion} \\ & & \quad\;\; \mid \; \widehat{x}P \cdot \alpha & \textit{right inversion} \end{array}$$

2. We extend the set of types by

$$A, B \;::=\; \cdots \mid \neg A$$

(as usual, $\neg A \rightarrow B$ stands for $(\neg A) \rightarrow B$).

3. We add the type assignment rules:

$$(\textit{r-inv}) : \dfrac{P \mathrel{:\cdot} \Gamma, x{:}A \vdash \Delta}{\widehat{x}P \cdot \alpha \mathrel{:\cdot} \Gamma \vdash \alpha{:}\neg A, \Delta} \qquad\qquad (\textit{l-inv}) : \dfrac{P \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta}{x \cdot P\widehat{\alpha} \mathrel{:\cdot} \Gamma, x{:}\neg A \vdash \Delta}$$

[14] An alternative way of treating negation is to add the type $\perp$, but in arrow types only on the right-hand side, and let *export* and *import* deal with it, but this would need separate constructs to introduce $\perp$ on either the right or the left; we feel our present approach is more clear.

Example 51 We can inhabit $\neg\neg A{\to}A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\langle y\cdot\alpha\rangle \mathrel{:\!\cdot} y{:}A \vdash \alpha{:}A}\,(\textit{cap})}{\widehat{y}\langle y\cdot\alpha\rangle\cdot\gamma \mathrel{:\!\cdot} {}\vdash \gamma{:}\neg A,\alpha{:}A}\,(\textit{r-inv})}{x\cdot(\widehat{y}\langle y\cdot\alpha\rangle\cdot\gamma)\widehat{\gamma} \mathrel{:\!\cdot} x{:}\neg\neg A\vdash\alpha{:}A}\,(\textit{l-inv})}{\widehat{x}\,(x\cdot(\widehat{y}\langle y\cdot\alpha\rangle\cdot\gamma)\widehat{\gamma})\widehat{\alpha}\cdot\beta \mathrel{:\!\cdot} {}\vdash\beta{:}\neg\neg A{\to}A}\,(\textit{exp})$$

The notion of reduction is extended naturally by adding the following rules.

Definition 52 We extend the notion of introduced connector by saying that also $P = x\cdot Q\widehat{\alpha}$ with $x \notin \mathit{fs}(Q)$ introduces x, and $P = \widehat{x}Q\cdot\alpha$ with $\alpha \notin \mathit{fp}(Q)$ introduces α.

The logical reduction rule for negation is (with β and x introduced):

$$(\widehat{y}P\cdot\beta)\widehat{\beta} \dagger \widehat{x}(x\cdot Q\widehat{\alpha}) \;\to\; Q\widehat{\alpha} \dagger \widehat{y}P$$

We add the propagation rules:

$$\begin{array}{lcll}
(y\cdot Q\widehat{\beta})\widehat{\alpha} \dagger \widehat{x}P & \to & y\cdot(Q\widehat{\alpha} \dagger \widehat{x}P)\widehat{\beta} & \\
(\widehat{y}Q\cdot\beta)\widehat{\alpha} \dagger \widehat{x}P & \to & \widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\cdot\beta & (\alpha \neq \beta) \\
(\widehat{y}Q\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}P & \to & (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\cdot\beta)\widehat{\beta} \dagger \widehat{x}P & (\beta\,\textit{fresh}, \alpha\ \textit{not introduced}) \\[1em]
P\widehat{\alpha} \dagger \widehat{x}(y\cdot Q\widehat{\beta}) & \to & y\cdot(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} & (x \neq y) \\
P\widehat{\alpha} \dagger \widehat{x}(x\cdot Q\widehat{\beta}) & \to & P\widehat{\alpha} \dagger \widehat{y}(y\cdot(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}) & (y\,\textit{fresh}, x\ \textit{not introduced}) \\
P\widehat{\alpha} \dagger \widehat{x}(\widehat{y}Q\cdot\beta) & \to & \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}Q)\cdot\beta &
\end{array}$$

and the contextual rules

$$P \to Q \;\Rightarrow\; \begin{cases} y\cdot P\widehat{\alpha} \;\to\; y\cdot Q\widehat{\alpha} \\ \widehat{y}P\cdot\alpha \;\to\; \widehat{y}Q\cdot\alpha \end{cases}$$

Notice that now we have *cuts* that do not contract, as

$$(\widehat{y}Q\widehat{\gamma}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(x\cdot P\widehat{\beta})$$

where $\alpha \notin \mathit{fp}(Q)$, and $x \notin \mathit{fs}(P)$, since none of the rules are applicable; however, *typeable cuts* do contract:

Theorem 53 *If* $P\widehat{\alpha} \dagger \widehat{x}Q \mathrel{:\!\cdot} \Gamma \vdash \Delta$, *then* $P\widehat{\alpha} \dagger \widehat{x}Q$ *can be contracted.*

Proof Easy. □

We can show:

Theorem 54 (Witness reduction) *If* $P : \Gamma \vdash_{\mathrm{LK}} \Delta$ *and* $P \to_{\pi} Q$, *then* $Q : \Gamma \vdash_{\mathrm{LK}} \Delta$.

Proof We just show the cases for some of the added rules; as mentioned above, the proof for the other rules can be found in [9].

$(\widehat{y}P\cdot\beta)\widehat{\beta} \dagger \widehat{x}(x\cdot Q\widehat{\alpha}) \to Q\widehat{\alpha} \dagger \widehat{y}P$: If $(\widehat{y}P\cdot\beta)\widehat{\beta} \dagger \widehat{x}(x\cdot Q\widehat{\alpha}) : \Gamma \vdash_{\text{LK}} \Delta$, then the derivation is shaped like:

$$\dfrac{\dfrac{P \mathrel{:\cdot} \Gamma, y{:}A \vdash \Delta}{\widehat{y}P\cdot\beta \mathrel{:\cdot} \Gamma \vdash \beta{:}\neg A, \Delta}\,(\textit{r-inv}) \qquad \dfrac{Q \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta}{x\cdot Q\widehat{\alpha} \mathrel{:\cdot} \Gamma, x{:}\neg A \vdash \Delta}\,(\textit{l-inv})}{(\widehat{y}P\cdot\beta)\widehat{\beta} \dagger \widehat{x}(x\cdot Q\widehat{\alpha}) \mathrel{:\cdot} \Gamma \vdash \Delta}\,(\textit{cut})$$

Then we can construct:

$$\dfrac{Q \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta \qquad P \mathrel{:\cdot} \Gamma, y{:}A \vdash \Delta}{Q\widehat{\beta} \dagger \widehat{x}P \mathrel{:\cdot} \Gamma \vdash \Delta}\,(\textit{cut})$$

$(\widehat{y}Q\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}P \to (\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\cdot\beta)\widehat{\beta} \dagger \widehat{x}P$ $(\beta\,\textit{fresh}, \alpha\ \textit{not introduced})$: The derivation for the left-hand side is shaped like:

$$\dfrac{\dfrac{Q \mathrel{:\cdot} \Gamma, y{:}A \vdash \alpha{:}\neg A, \Delta}{\widehat{y}Q\cdot\alpha \mathrel{:\cdot} \Gamma \vdash \alpha{:}\neg A, \Delta}\,(\textit{r-inv}) \qquad P \mathrel{:\cdot} \Gamma, x{:}\neg A \vdash \Delta}{(\widehat{y}Q\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}P \mathrel{:\cdot} \Gamma \vdash \Delta}\,(\textit{cut})$$

Then we can construct:

$$\dfrac{\dfrac{\dfrac{Q \mathrel{:\cdot} \Gamma, y{:}A \vdash \alpha{:}\neg A, \Delta \qquad \dfrac{P, x{:}\neg A \mathrel{:\cdot} \Gamma \vdash \Delta}{P \mathrel{:\cdot} \Gamma, x{:}\neg A, y{:}A \vdash \Delta}\,(W)}{Q\widehat{\alpha} \dagger \widehat{x}P \mathrel{:\cdot} \Gamma, y{:}A \vdash \Delta}\,(\textit{cut})}{\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\cdot\beta \mathrel{:\cdot} \Gamma \vdash \beta{:}\neg A, \Delta}\,(\textit{r-inv}) \qquad P \mathrel{:\cdot} \Gamma, x{:}\neg A \vdash \Delta}{(\widehat{y}(Q\widehat{\alpha} \dagger \widehat{x}P)\cdot\beta)\widehat{\beta} \dagger \widehat{x}P \mathrel{:\cdot} \Gamma \vdash \Delta}\,(\textit{cut})$$

$P\widehat{\alpha} \dagger \widehat{x}(y\cdot Q\widehat{\beta}) \to y\cdot(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}$ $(x \neq y)$: The derivation for the left-hand side is shaped like:

$$\dfrac{P \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \alpha{:}A, \Delta \qquad \dfrac{Q \mathrel{:\cdot} \Gamma, x{:}A \vdash \beta{:}B, \Delta}{y\cdot Q\widehat{\beta} \mathrel{:\cdot} \Gamma, x{:}A, y{:}\neg B \vdash \Delta}\,(\textit{l-inv})}{P\widehat{\alpha} \dagger \widehat{x}(y\cdot Q\widehat{\beta}) \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \Delta}\,(\textit{cut})$$

and we can construct:

$$\dfrac{\dfrac{\dfrac{P \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \alpha{:}A, \Delta}{P \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \alpha{:}A, \beta{:}B, \Delta}\,(W) \qquad \dfrac{Q \mathrel{:\cdot} \Gamma, x{:}A \vdash \beta{:}B, \Delta}{Q \mathrel{:\cdot} \Gamma, y{:}\neg B, x{:}A \vdash \beta{:}B, \Delta}\,(W)}{P\widehat{\alpha} \dagger \widehat{x}Q \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \beta{:}B, \Delta}\,(\textit{cut})}{y\cdot(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \mathrel{:\cdot} \Gamma, y{:}\neg B \vdash \Delta}\,(\textit{l-inv})$$

The other cases are similar. □

We will now extend the two translations so that we deal with the added connective as well.

Definition 55 (Negation) Negation gets represented in the π-calculus via the natural translation as:

$$\begin{array}{rcl} \llbracket x\cdot P\widehat{\alpha}\rrbracket_{\mathrm{N}} & = & x(\alpha).!\,\llbracket P\rrbracket_{\mathrm{N}} \\ \llbracket \widehat{x}P\cdot\alpha\rrbracket_{\mathrm{N}} & = & (\nu x)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{\alpha}\langle x\rangle) \end{array}$$

via the semantic translation as:

$$\begin{array}{rcl} \llbracket x\cdot P\widehat{\alpha}\rrbracket_{\mathrm{S}} & = & (\nu\alpha)\,(\llbracket P\rrbracket_{\mathrm{S}} \mid !\,x(z).!\alpha{\to}z) \\ \llbracket \widehat{x}P\cdot\alpha\rrbracket_{\mathrm{S}} & = & (\nu x)\,(\llbracket P\rrbracket_{\mathrm{S}} \mid !\overline{\alpha}\langle x\rangle) \end{array}$$

This translation of inversion explains the role of negation in detail. If P is outputting on α, but no connection to α is available, input is needed from a process Q that will send one of its input names z. Once received, P can output on α which gets connected to z; so Q will provide a means for P to continue, and is therefore aptly called a *continuation*.

The natural and semantic translations of the witness for $\neg\neg A{\to}A$ now become:

$$\llbracket \widehat{x}\,(x\cdot(\widehat{y}\langle y\cdot\alpha\rangle\cdot\gamma)\,\widehat{\gamma})\widehat{\alpha}\cdot\beta\rrbracket_{\mathrm{N}} \;=\; (\nu x\alpha)\,(!\,x(\gamma).!\,(\nu y)\,(!\,y(w).\overline{\alpha}\langle w\rangle \mid \overline{\gamma}\langle y\rangle) \mid \overline{\beta}\langle x,\alpha\rangle)$$

$$\begin{array}{l} \llbracket \widehat{x}\,(x\cdot(\widehat{y}\langle y\cdot\alpha\rangle\cdot\gamma)\,\widehat{\gamma})\widehat{\alpha}\cdot\beta\rrbracket_{\mathrm{S}} \;= \\ \qquad\qquad (\nu x\alpha)\,((\nu\gamma)\,((\nu y)\,(!\,y(w).\overline{\alpha}\langle w\rangle \mid !\,\overline{\gamma}\langle y\rangle) \mid !\,x(z).!\,\gamma{\to}z) \mid !\,\overline{\beta}\langle x,\alpha\rangle) \end{array}$$

The following consistency results are easy to prove.

Theorem 56 *Let* $(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})$ *be such that* β, x *are introduced. Then*

1. $\llbracket\,(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})\rrbracket_{\mathrm{N}} \;\mathrel{\underset{\sim}{\sqsupset}}_{\pi}\; \llbracket Q\widehat{\alpha}\dagger\widehat{y}P\rrbracket_{\mathrm{N}}$.

2. $\llbracket\,(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})\rrbracket_{\mathrm{S}} \;\approx\; \llbracket Q\widehat{\alpha}\dagger\widehat{y}P\rrbracket_{\mathrm{S}}$.

Proof 1.

$$\begin{array}{ll} \llbracket\,(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})\rrbracket_{\mathrm{N}} & \triangleq \qquad . \\ (\nu x)\,(!\,(\nu y)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y\rangle) \mid !\,x(\alpha).!\,\llbracket Q\rrbracket_{\mathrm{N}}) & \to_{\pi}\,(x) \\ (\nu y)\,(!\,\llbracket Q[y/\alpha]\rrbracket_{\mathrm{N}} \mid !\,\llbracket P\rrbracket_{\mathrm{N}}) \mid (\nu x)\,(!\,(\nu y)\,(!\,\llbracket P\rrbracket_{\mathrm{N}} \mid \overline{x}\langle y\rangle) \mid !\,x(\alpha).!\,\llbracket Q\rrbracket_{\mathrm{N}}) & \triangleq \\ \llbracket Q\widehat{\alpha}\dagger\widehat{y}P\rrbracket_{\mathrm{N}} \mid \llbracket\,(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})\rrbracket_{\mathrm{N}} & \mathrel{\underset{\sim}{\sqsupset}}_{\pi}\;\llbracket Q\widehat{\alpha}\dagger\widehat{y}P\rrbracket_{\mathrm{N}} \end{array}$$

2.

$$\begin{array}{ll} \llbracket\,(\widehat{y}P\cdot\beta)\,\widehat{\beta}\dagger\widehat{x}\,(x\cdot Q\widehat{\alpha})\rrbracket_{\mathrm{S}} & \triangleq \\ (\nu\beta x)\,((\nu y)\,(\llbracket P\rrbracket_{\mathrm{S}} \mid !\,\overline{\beta}\langle y\rangle) \mid !\,\beta{\to}x \mid (\nu\alpha)\,(\llbracket Q\rrbracket_{\mathrm{S}} \mid !\,x(z).!\,\alpha{\to}z)) & \approx(\beta,x) \\ (\nu\Box\alpha y)\,(\llbracket Q\rrbracket_{\mathrm{S}} \mid !\,\alpha{\to}y \mid \llbracket P\rrbracket_{\mathrm{S}}) & \triangleq\;\llbracket Q\widehat{\alpha}\dagger\widehat{y}P\rrbracket_{\mathrm{S}} \end{array}$$

We add the following type assignment rules for negation:

Definition 57 (Type assignment rules in $\vdash_{\pi}$ for $\neg$)

$$(\mathit{inv\text{-}r}):\;\frac{}{\overline{a}\langle x\rangle : x{:}A \vdash a{:}\neg A} \qquad\qquad (\mathit{inv\text{-}l}):\;\frac{P : \Gamma\vdash x{:}A,\Delta}{a(x).P : \Gamma, a{:}\neg A\vdash\Delta}\;(a\notin\Gamma)$$

The correctness of the propagation rules follows as above in Theorems 28 and 35; notice that, since negation gets interpreted in the natural translation using input, the first contextual rule has to be excluded from $\to_{\mathrm{H}}$.

We can now check that the extended translation preserves assignable types as well.

Theorem 58 *If* $P \mathrel{:\!\cdot}\, \Gamma\vdash\Delta$, *then* $\llbracket P\rrbracket_{\mathrm{N}} : \Gamma\vdash_{\pi}\Delta$.

Proof By induction on the structure of of terms in $\mathcal{LK}$; we only show the two added cases.

$x\cdot P\widehat{\alpha}$: Then the last rule applied in the $\mathcal{LK}$-derivation is $(\textit{l-inv})$:

$$\dfrac{P \;\therefore\; \Gamma \vdash \alpha{:}A, \Delta}{x\cdot P\widehat{\alpha} \;\therefore\; \Gamma, x{:}\neg A \vdash \Delta}\;(\textit{l-inv})$$

and, by induction, $[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A, \Delta$, and we can construct:

$$\dfrac{\dfrac{[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A, \Delta}{!\,[\![P]\!]_{\mathrm{N}} : \Gamma \vdash_\pi \alpha{:}A, \Delta}\;(!)}{x(\alpha).!\,[\![P]\!]_{\mathrm{N}} : \Gamma, x{:}\neg A \vdash_\pi \Delta}\;(\textit{inv-l})$$

and $[\![x\cdot P\widehat{\alpha}]\!]_{\mathrm{N}} = x(\alpha).!\,[\![P]\!]_{\mathrm{N}}$.

$\widehat{x}P\cdot\alpha$: Then the derivation is shaped like:

$$\dfrac{P \;\therefore\; \Gamma, x{:}A \vdash \Delta}{\widehat{x}P\cdot\alpha \;\therefore\; \Gamma \vdash \alpha{:}\neg A, \Delta}\;(\textit{r-inv})$$

and, by induction, $[\![P]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta$, and we can construct:

$$\dfrac{\dfrac{\dfrac{[\![P]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta}{!\,[\![P]\!]_{\mathrm{N}} : \Gamma, x{:}A \vdash_\pi \Delta}\;(!) \qquad \dfrac{}{\overline{\alpha}\langle x\rangle : x{:}A \vdash_\pi \alpha{:}\neg A}\;(\textit{inv-r})}{!\,[\![P]\!]_{\mathrm{N}} \mid \overline{\alpha}\langle x\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}\neg A, \Delta}\;(\mid)}{(\nu x)\,(!\,[\![P]\!]_{\mathrm{N}} \mid \overline{\alpha}\langle x\rangle) : \Gamma \vdash_\pi \alpha{:}\neg A, \Delta}\;(\nu)$$

and $[\![\widehat{x}P\cdot\alpha]\!]_{\mathrm{N}} = (\nu x)\,(!\,[\![P]\!]_{\mathrm{N}} \mid \overline{\alpha}\langle x\rangle)$. $\square$

Similarly, we can show:

Theorem 59 *If $P \;\therefore\; \Gamma \vdash \Delta$, then $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \Delta$.*

Proof By induction on the structure of of terms in $\mathcal{LK}$; as above, we only show the two added cases to the proof of Theorem 44.

$x\cdot P\widehat{\alpha}$: Then the derivation is shaped like:

$$\dfrac{P \;\therefore\; \Gamma \vdash \alpha{:}A, \Delta}{x\cdot P\widehat{\alpha} \;\therefore\; \Gamma, x{:}\neg A \vdash \Delta}\;(\textit{l-inv})$$

and, by induction, $[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A, \Delta$, and we can construct:

$$\dfrac{\dfrac{[\![P]\!]_{\mathrm{S}} : \Gamma \vdash_\pi \alpha{:}A, \Delta \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{z}\langle w\rangle : w{:}A \vdash_\pi z{:}A, w{:}A}\;(\textit{out})}{\alpha\to z : \alpha{:}A \vdash_\pi z{:}A}\;(\textit{in})}{!\,\alpha\to z : \alpha{:}A \vdash_\pi z{:}A}\;(!)}{x(z).(!\,\alpha\to z) : \alpha{:}A, x{:}\neg A \vdash_\pi}\;(\textit{inv-l})}{!\,x(z).(!\,\alpha\to z) : \alpha{:}A, x{:}\neg A \vdash_\pi}\;(!)}{[\![P]\!]_{\mathrm{S}} \mid !\,x(z).(!\,\alpha\to z) : \Gamma, \alpha{:}A, x{:}\neg A \vdash_\pi \alpha{:}A, \Delta}\;(\mid)}{(\nu\alpha)\,([\![P]\!]_{\mathrm{S}} \mid !\,x(z).!\,\alpha\to z) : \Gamma, x{:}\neg A \vdash_\pi \Delta}\;(\nu)$$

and $\llbracket x\cdot P\widehat{\alpha}\rrbracket_{\mathrm{s}} = (\nu\alpha)\,(\llbracket P\rrbracket_{\mathrm{s}} \mid {!}x(z).{!}\alpha{\rightarrow}z)$.

$\widehat{x}P\cdot\alpha$: Then the derivation is shaped like:

$$\dfrac{\begin{array}{c}\boxed{}\\ P \mathrel{:\cdot} \Gamma, x{:}A \vdash \Delta\end{array}}{\widehat{x}P\cdot\alpha \mathrel{:\cdot} \Gamma \vdash \alpha{:}\neg A, \Delta}\;(\textit{r-inv})$$

and, by induction, $\llbracket P\rrbracket_{\mathrm{s}} : \Gamma, x{:}A \vdash_\pi \Delta$, and we can construct:

$$\dfrac{\dfrac{\begin{array}{c}\boxed{}\\ \llbracket P\rrbracket_{\mathrm{s}} : \Gamma, x{:}A \vdash_\pi \Delta\end{array} \qquad \dfrac{\dfrac{}{\overline{\alpha}\langle x\rangle : x{:}A \vdash_\pi \alpha{:}\neg A}\;(\textit{out})}{{!}\overline{\alpha}\langle x\rangle : x{:}A \vdash_\pi \alpha{:}\neg A}\;(\textit{inv-r})}{\llbracket P\rrbracket_{\mathrm{s}} \mid {!}\overline{\alpha}\langle x\rangle : \Gamma, x{:}A \vdash_\pi \alpha{:}\neg A, \Delta}\;(\mid)}{(\nu x)\,(\llbracket P\rrbracket_{\mathrm{s}} \mid {!}\overline{\alpha}\langle x\rangle) : \Gamma \vdash_\pi \alpha{:}\neg A, \Delta}\;(\nu)$$

and $\llbracket \widehat{x}P\cdot\alpha\rrbracket_{\mathrm{s}} = (\nu x)\,(\llbracket P\rrbracket_{\mathrm{s}} \mid {!}\overline{\alpha}\langle x\rangle)$. □

So our extended translations respect the classical sequent logic rules.

Conclusions

In this paper we have bridged the gap between classical *cut*-elimination and the semantics of concurrent calculi, by presenting translations of Gentzen's classical sequent calculus LK to the π-calculus that preserve *cut*-elimination. This was achieved through an embedding of the calculus $\mathcal{LK}$ into the π-calculus that translates a *cut* as synchronisation. $\mathcal{LK}$'s terms directly represent proofs in LK, by naming assumptions with Roman characters, and conclusions with Greek characters, and seeing these as *input* and *output*, respectively, but terms in $\mathcal{LK}$ can also not correspond to proofs.

$\mathcal{LK}$ introduces a natural concept of input and output that naturally translates into the input and output primitives of the π-calculus. We presented two different translations, each with specific interesting properties. We first presented the natural translation, and showed that it preserves $\mathcal{LK}$'s head-reduction; in this translation we cannot represent full *cut*-elimination because we place some interpreted terms under *input*, in particular when interpreting the witness for $(\rightarrow L)$. This seems to be a natural consequence, and is a feature also in the various translations of the λ-calculus.

We then went on to show that the limitation of *input* can easily be avoided. To that purpose, we introduced the concept of *synchronisation cell*, and managed to show that, by slightly modifying our translation and interpreting terms as infinite resources, we can represent full *cut*-elimination, but not through synchronisation, but rather weak bisimilarity. We have seen that this translation successfully represents Gentzen's *Hauptsatz* result, in that innermost reduction on typeable terms terminates.

The variant of the π-calculus we considered uses a pairing facility which enables the definition of a notion of implicative type assignment on processes. Using this notion, we proved that proofs in LK have a representation in π; our *cut*-elimination results then show that not only do we correctly represent reduction on the calculus $\mathcal{LK}$, but also can model proofs in LK in all detail in such a way that *cut*-elimination is preserved by weak bisimilarity. We also represented negation in $\mathcal{LK}$ by extending the syntax and reduction rules, and extended our translations to deal with the added construct. We have shown that all representation results still hold; since we have successfully represented both implication and negation, this implies that this can then easily be extended to the other logical connectives.

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