Draft

Notes about $F^{\omega}_{<:}$

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$F^{\omega}_{<:}$

This is the generalization of $F_{<:}$ to ω-order, plus monotonicity.

K ::=	Kinds
TY	the kind of all types
$\mathcal{P}_K(A)$	the kind of all subfamilies of a family
$\Pi(X::K)K'$	the kind of operators between kinds
$\Pi^+(X::K)K'$	the kind of monotonic operators
$\Pi^-(X::K)K'$	the kind of antimonotonic operators

A ::=	Families (Types and Operators)
X	type variables
Top	the supertype of all types
$A \rightarrow A'$	function spaces
$\forall(X::K)A'$	bounded quantifications
$\Lambda(X::K)A$	operators
A(A')	operator applications

a ::=	Values
x	value variables
top	canonical value of type Top
$\lambda(x:A)a$	functions
a(a')	applications
$\lambda(X::K)a$	bounded type functions
a(A)	type applications

Environments

In $F_{<:}$ $\quad \varnothing \quad E,x:A \quad E,X<:A$

In $F^{\omega}_{<:}$ $\quad \varnothing \quad E,x:A \quad E,X::K$

Judgments

In $F_{<:}$ $\quad E \vdash A \text{ type} \equiv E \vdash A :: TY$

$\quad E \vdash A <: B \equiv E \vdash A :: \mathcal{P}_{TY}(B)$

In $F^{\omega}_{<:}$ $\quad E \vdash A :: K$

Notation

$A<:B::K \equiv A::\mathcal{P}_K(B)$

Note

$TY \approx \mathcal{P}_{TY}(Top)$

Judgments

$\vdash E$

$E \vdash K\ kind$

$E \vdash A::K$

$E \vdash a:A$

$E \vdash K<::L$

$E \vdash A<:B :: K$

$E \vdash A=B :: K$

$E \vdash a=b : A$

Rules

$$\frac{\vdash E, X::\mathcal{P}_K(A), E' \text{ env}}{E, X::\mathcal{P}_K(A), E' \vdash X::K} \qquad \frac{\vdash E, X::K, E' \text{ env}}{E, X::K, E' \vdash X::K}$$

$$(\Pi\ I)\ \frac{E, X::K \vdash B::L}{E \vdash \Lambda(X::K)B :: \Pi(X::K)L} \qquad (\Pi\ E)\ \frac{E \vdash B::\Pi(X::K)L \quad E \vdash A::K}{E \vdash B(A)::L\{X \leftarrow A\}}$$

$$(\Pi^+ I)\ \frac{E \vdash B::\Pi(X::K)L \quad E, Y'::K, Y<:Y'::K \vdash B(Y)<:B(Y')::L}{E \vdash B :: \Pi^+(X::K)L} \qquad (\Pi^+ E)\ \frac{E \vdash B::\Pi^+(X::K)L \quad E \vdash A::K}{E \vdash B(A) :: L\{X \leftarrow A\}}$$

$$(\Pi^- I)\ \frac{E \vdash B::\Pi(X::K)L \quad E, Y'::K, Y<:Y'::K \vdash B(Y')<:B(Y)::L}{E \vdash B :: \Pi^-(X::K)L} \qquad (\Pi^- E)\ \frac{E \vdash B::\Pi^-(X::K)L \quad E \vdash A::K}{E \vdash B(A) :: L\{X \leftarrow A\}}$$

$$(\Pi <: I)\ \frac{E, X::K \vdash B<:B'::L}{E \vdash \Lambda(X::K)B <: \Lambda(X::K)B' :: \Pi(X::K)L} \qquad (\Pi <: E)\ \frac{E \vdash B<:B'::\Pi(X::K)L \quad E \vdash A::K}{E \vdash B(A) <: B'(A) :: L\{X \leftarrow A\}}$$

$$(\Pi^+ <: I)\ \frac{E \vdash B<:B'::\Pi(X::K)L \quad E \vdash B :: \Pi^+(X::K)L \quad E \vdash B' :: \Pi^+(X::K)L}{E \vdash B <: B' :: \Pi^+(X::K)L} \qquad (\Pi^+ <: E)\ \frac{E \vdash B::\Pi^+(X::K)L \quad E \vdash A<:A'::K}{E \vdash B(A) <: B(A') :: L\{X \leftarrow A\}}$$

$$(\Pi^- <: I)\ \frac{E \vdash B<:B'::\Pi(X::K)L \quad E \vdash B :: \Pi^-(X::K)L \quad E \vdash B' :: \Pi^-(X::K)L}{E \vdash B <: B' :: \Pi^-(X::K)L} \qquad (\Pi^- <: E)\ \frac{E \vdash B::\Pi^-(X::K)L \quad E \vdash A<:A'::K}{E \vdash B(A') <: B(A) :: L\{X \leftarrow A\}}$$

---- beta, eta

3. Operator fragment

(A simplified version of the previous rules where $X \notin FV(L)$, etc.)

$$\frac{\vdash E, X<:A::K, E' \text{ env}}{E, X<:A::K, E' \vdash X::K} \qquad \frac{\vdash E, X<:A::K, E' \text{ env}}{E, X<:A::K, E' \vdash X<:A::K}$$

(⇒ I)
$$\frac{E, X::K \vdash B::L}{E \vdash \Lambda(X::K)B :: K \Rightarrow L}$$

(⇒ E)
$$\frac{E \vdash B::K \Rightarrow L \quad E \vdash A::K}{E \vdash B(A)::L}$$

($\Rightarrow^+$ I)
$$\frac{E \vdash B::K \Rightarrow L \quad E, Y'::K, Y<:Y'::K \vdash B(Y)<:B(Y')::L}{E \vdash B :: K \Rightarrow^+ L}$$

($\Rightarrow^+$ E)
$$\frac{E \vdash B::K \Rightarrow^+ L \quad E \vdash A::K}{E \vdash B(A) :: L}$$

($\Rightarrow^-$ I)
$$\frac{E \vdash B::K \Rightarrow L \quad E, Y'::K, Y<:Y'::K \vdash B(Y')<:B(Y)::L}{E \vdash B :: K \Rightarrow^- L}$$

($\Rightarrow^-$ E)
$$\frac{E \vdash B::K \Rightarrow^- L \quad E \vdash A::K}{E \vdash B(A) :: L}$$

(⇒ <: I)
$$\frac{E, X::K \vdash B<:B'::L}{E \vdash \Lambda(X::K)B <: \Lambda(X::K)B' :: K \Rightarrow L}$$

(⇒ <: E)
$$\frac{E \vdash B<:B'::K \Rightarrow L \quad E \vdash A::K}{E \vdash B(A) <: B'(A) :: L}$$

($\Rightarrow^+$ <: I)
$$\frac{E \vdash B<:B'::K \Rightarrow L \quad E \vdash B :: K \Rightarrow^+ L \quad E \vdash B' :: K \Rightarrow^+ L}{E \vdash B <: B' :: K \Rightarrow^+ L}$$

($\Rightarrow^+$ <: E)
$$\frac{E \vdash B::K \Rightarrow^+ L \quad E \vdash A<:A'::K}{E \vdash B(A) <: B(A') :: L}$$

($\Rightarrow^-$ <: I)
$$\frac{E \vdash B<:B'::K \Rightarrow L \quad E \vdash B :: K \Rightarrow^- L \quad E \vdash B' :: K \Rightarrow^- L}{E \vdash B <: B' :: K \Rightarrow^- L}$$

($\Rightarrow^-$ <: E)
$$\frac{E \vdash B::K \Rightarrow^- L \quad E \vdash A<:A'::K}{E \vdash B(A') <: B(A) :: L}$$

(beta)
$$\frac{E, X::K \vdash B = B' :: L \quad E \vdash A = A' :: K}{E \vdash (\Lambda(X::K)B)(A) = B'\{X \leftarrow A'\} :: L}$$

(eta)
$$\frac{E \vdash B = B'::K \Rightarrow L \quad X \notin dom(E)}{E \vdash \Lambda(X::K)B(X) = B' :: K \Rightarrow L}$$

4. Lemmas

Prop

$$\frac{E \vdash F::K \Rightarrow^{+} (L \Rightarrow M) \quad E \vdash F::K \Rightarrow (L \Rightarrow^{+} M)}{E \vdash F::K \Rightarrow^{+} (L \Rightarrow^{+} M)}$$

Proof

$$\frac{E \vdash F::K \Rightarrow (L \Rightarrow^{+} M) \qquad E \vdash F::K \Rightarrow (L \Rightarrow^{+} M)}{1)\ E,Y'::K,Y<:Y'::K \vdash F(Y)::L \Rightarrow^{+} M \quad E,Y'::K,Y<:Y'::K \vdash F(Y')::L \Rightarrow^{+} M}\ \text{(weaken) } (\Rightarrow E)$$

$$\frac{\dfrac{\dfrac{E \vdash F::K \Rightarrow^{+} (L \Rightarrow M)}{E,Y'::K,Y<:Y'::K \vdash F(Y)<:F(Y')::L \Rightarrow M}\ \text{(weaken) } (\Rightarrow^{+} <: E) \qquad (1)}{E,Y'::K,Y<:Y'::K \vdash F(Y)<:F(Y')::L \Rightarrow^{+} M}\ (\Rightarrow^{+} <: I) \qquad E \vdash F::K \Rightarrow (L \Rightarrow^{+} M)}{E \vdash E \vdash F::K \Rightarrow^{+} (L \Rightarrow^{+} M)}\ (\Rightarrow^{+} I)$$

Prop

$$\frac{\begin{array}{c} E \vdash B::K \Rightarrow L \Rightarrow M \\ E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y')(Z)::M \\ E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y)(Z')::M \end{array}}{E \vdash B :: K \Rightarrow^{+} L \Rightarrow^{+} M}\ (\Rightarrow^{+} I\ 2)$$

Proof

$$\frac{\dfrac{\dfrac{E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y')(Z)::M}{E,Y'::K,Y<:Y'::K \vdash \lambda(Z::L)B(Y)(Z)<:\lambda(Z::L)B(Y')(Z)::L \Rightarrow M}}{E,Y'::K,Y<:Y'::K \vdash B(Y)<:B(Y')::L \Rightarrow M}}{E \vdash B::K \Rightarrow^{+} L \Rightarrow M}$$

$$\frac{\dfrac{\dfrac{\dfrac{E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y)(Z')::M}{E,Y'::K,Y<:Y'::K \vdash B(Y)::L \Rightarrow^{+} M}}{E,Y'::K \vdash \lambda(Y<:Y'::K)B(Y)::\Pi(Y<:Y'::K)L \Rightarrow^{+} M}}{E,Y'::K \vdash B(Y')::L \Rightarrow^{+} M}}{E \vdash \lambda(Y'::K)B(Y')::K \Rightarrow L \Rightarrow^{+} M}}{E \vdash B::K \Rightarrow L \Rightarrow^{+} M}$$

Hence $E \vdash B :: K \Rightarrow^{+} L \Rightarrow^{+} M$ by the previous proposition.

5. The kind NAT in $F^{\omega}_{<:}$

By analogy with Church numerals:

NAT' kind $\triangleq$ $(TY \Rightarrow TY) \Rightarrow (TY \Rightarrow TY)$

Zero :: NAT' $\triangleq$ $\Lambda(S::TY \Rightarrow TY)\, \Lambda(Z::TY)\, Z$
Succ :: NAT' $\Rightarrow$ NAT' $\triangleq$
$\Lambda(N::NAT)\, \Lambda(S::TY \Rightarrow TY)\, \Lambda(Z::TY)\, S(N(S)(Z))$

Suppose now we want Succ to be monotonic. We are forced into the following definition, where all $\Rightarrow^+$ are necessary:

NAT kind $\triangleq$ $(TY \Rightarrow^+ TY) \Rightarrow^+ (TY \Rightarrow^+ TY)$

Zero :: NAT' $\triangleq$ $\Lambda(S::TY \Rightarrow^+ TY)\, \Lambda(Z::TY)\, Z$
Succ :: NAT $\Rightarrow$ NAT' $\triangleq$
$\Lambda(N::NAT)\, \Lambda(S::TY \Rightarrow^+ TY)\, \Lambda(Z::TY)\, S(N(S)(Z))$

It is easy to show that:
Zero :: NAT
Succ :: NAT $\Rightarrow$ NAT

The we can prove:

$A <: B::NAT \supset A(S)(Z) <: B(S)(Z)::TY$ for $S::TY \Rightarrow^+ TY$, $Z::TY$
$\supset \Lambda(Z::TY)A(S)(Z) <: \Lambda(Z::TY)B(S)(Z) :: TY \Rightarrow^+ TY$
$\supset \Lambda(S::TY \Rightarrow^+ TY)\Lambda(Z::TY)A(S)(Z) <:$
$\Lambda(S::TY \Rightarrow^+ TY)\Lambda(Z::TY)B(S)(Z) :: NAT$
$\supset Succ(A) <: Succ(B) :: NAT$

That is, Succ :: NAT $\Rightarrow^+$ NAT

Morale: in order to obtain monotonic constructors, the iterator of kind NAT must be monotonic in all positions.

6. Pairs of types in $F^{\omega}_{<:}$

PAIR' kind $\triangleq$ $(TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow TY$

PAIR kind $\triangleq$ $(TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow^{+} TY$

Pair :: $TY \Rightarrow TY \Rightarrow PAIR'$
$\triangleq$ $\Lambda(X::TY)\ \Lambda(Y::TY)\ \Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(X)(Y)$

Prop: Pair :: $TY \Rightarrow TY \Rightarrow PAIR$
Proof
Assume A,B::TY
Assume $Y <: Y' :: TY \Rightarrow^{+} TY \Rightarrow^{+} TY$
Then $Y(A) <: Y'(A) :: TY \Rightarrow^{+} TY$ $(\Rightarrow^{+} <: E)$
Then $Y(A)(B) <: Y'(A)(B) :: TY$ $(\Rightarrow^{+} <: E)$
Then $(\Lambda(X::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ X(A)(B))(Y)$
$<: (\Lambda(X::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ X(A)(B))(Y') :: TY$ (β)
I.e.: $Pair(A)(B)(Y) <: Pair(A)(B)(Y') :: TY$
Hence $Pair(A)(B) :: (TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow^{+} TY = PAIR$ $(\Rightarrow^{+} I)$
Finally $\Lambda(A::TY)\ \Lambda(B::TY)\ Pair(A)(B) :: TY \Rightarrow TY \Rightarrow PAIR$ $(\Rightarrow I)$
and $Pair :: TY \Rightarrow TY \Rightarrow PAIR$ (η)

Prop: Pair :: $TY \Rightarrow TY \Rightarrow^{+} PAIR$
Proof
Assume A::TY, B<:B'
Assume $W::TY \Rightarrow^{+} (TY \Rightarrow^{+} TY)$
Hence $W(A)(B) <: W(A)(B') :: TY$ $(\Rightarrow^{+} <: E)$
Hence $(\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A)(B))(W)$
$<: (\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A)(B'))(W) :: TY$ (β)
I.e.: $Pair(A)(B)(W) <: Pair(A)(B')(W) :: TY$
Then $\Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A)(B)(W)$
$<: \Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A)(B')(W) :: PAIR'$ $(\Rightarrow <: I)$
I.e.: $Pair(A)(B) <: Pair(A)(B') :: PAIR'$ (η)
But Pair(A)(B),Pair(A)(B') :: PAIR (previous prop)
Hence $Pair(A)(B) <: Pair(A)(B') :: PAIR$ $(\Rightarrow^{+} <: I)$
Finally $Pair(A) :: TY \Rightarrow^{+} PAIR$ $(\Rightarrow^{+} I)$
and $Pair :: TY \Rightarrow TY \Rightarrow^{+} PAIR$ $(\Rightarrow I)\ (\eta)$

Prop: Pair :: TY⇒⁺TY⇒⁺PAIR
Proof
Assume A<:A', B::TY
Assume W::TY⇒⁺TY⇒⁺TY
Hence W(A)(B) <: W(A')(B) :: TY (⇒⁺ <: E)
Hence (Λ(Z::TY⇒⁺TY⇒⁺TY) Z(A)(B))(W)
<: (Λ(Z::TY⇒⁺TY⇒⁺TY) Z(A')(B))(W) :: TY (β)
I.e.: Pair(A)(B)(W) <: Pair(A')(B)(W) :: TY
Then Λ(W::TY⇒⁺TY⇒⁺TY)Pair(A)(B)(W)
<: Λ(W::TY⇒⁺TY⇒⁺TY)Pair(A')(B)(W) :: PAIR' (⇒ <: I)
I.e.: Pair(A)(B) <: Pair(A')(B) :: PAIR' (η)
But Pair(A)(B),Pair(A')(B) :: PAIR (previous prop)
Hence Pair(A)(B) <: Pair(A')(B) :: PAIR (⇒⁺ <: I)
Then Λ(B::TY)Pair(A)(B)
<: Λ(B::TY)Pair(A')(B) :: TY⇒PAIR (⇒ <: I)
I.e.: Pair(A) <: Pair(A') :: TY⇒PAIR (η)
But Pair(A),Pair(A') :: TY⇒⁺PAIR (previous prop)
Hence Pair(A) <: Pair(A') :: TY⇒⁺PAIR (⇒⁺ <: I)
Finally Pair <: Pair :: TY⇒⁺TY⇒⁺PAIR (⇒⁺ I)

\+ Cor: If A<:A', B<:B' then Pair(A)(B) <: Pair(A')(B') :: PAIR
Proof From Pair :: TY⇒⁺TY⇒⁺PAIR by (⇒⁺ <: E)

Fst :: PAIR⇒TY
≜ Λ(P::PAIR) P(Λ(X::TY) Λ(Y::TY) X)

Prop: Fst :: PAIR⇒⁺TY
(If P<:P'::PAIR then Fst(P)<:Fst(P'))
Proof
Assume P <: P' :: PAIR
Note that Λ(X::TY) Λ(Y::TY) X :: TY⇒⁺TY⇒⁺TY
Then P(Λ(X::TY) Λ(Y::TY) X)
<: P'(Λ(X::TY) Λ(Y::TY) X) :: TY (⇒⁺ <: E)
Then (Λ(P::PAIR) P(Λ(X::TY) Λ(Y::TY) X))(P)
<: (Λ(P::PAIR) P(Λ(X::TY) Λ(Y::TY) X))(P') :: TY (β)
I.e.: Fst(P) <: Fst(P') :: TY
Hence Fst :: PAIR⇒⁺TY (⇒⁺ I)

Snd :: PAIR⇒TY
≜ Λ(P::PAIR) P(Λ(X::TY) Λ(Y::TY) Y)

Prop: Snd :: PAIR⇒⁺TY
(If P<:P'::PAIR then Snd(P)<:Snd(P'))

Prod :: PAIR⇒TY
≜ Λ(P::PAIR) Fst(P)×Snd(P)

Prop: Prod :: PAIR⇒⁺TY
(If P<:P'::PAIR then Prod(P) <: Prod(P') :: TY)
Proof
Assume Q<:Q'::PAIR
By hyp and previous facts: Fst(Q) <: Fst(Q'); Snd(Q) <: Snd(Q')
By monotonicity of ×: Fst(Q)×Snd(Q) <: Fst(Q')×Snd(Q')
Expansion: (Λ(P::PAIR) Fst(P)×Snd(P))(Q)
<: (Λ(P::PAIR) Fst(P)×Snd(P))(Q')
I.e.: Prod(Q) <: Prod(Q')
Hence Prod :: PAIR⇒⁺TY (⇒⁺ I)