$\mathcal{F}_{w<:}$

Syntax

K, L ::=	Kinds
Ty | Π(X<:A::K)L	
A,B ::=	Constructors
X | Top(K) | λ(X<:A::K)B | B(A) | A→B | ∀(X<:A::K)B | μ(X)A	
a,b ::=	Values
x | λ(x:A)b | b(a) | λ(X<:A::K)b | b(A') |	
fold(μ(X)B{X},a) | unfold(a) | μ(x:A)b	

Judgments

E ⊢ ◇	E is an environment
E ⊢ K kind	K is a kind
E ⊢ K <:: L kind	K is a subkind of L (optional)
E ⊢ K ↔ L kind	K and L are equivalent kinds
E ⊢ A :: K	constructor A has kind K
E ⊢ A <: B :: K	A is a subconstructor of B, both of kind K
E ⊢ A ↔ B :: K	A and B are equivalent constructors of kind K
E ⊢ a : A	value a has type A
E ⊢ a ↔ b : A	a and b are equivalent values of type A

Abbreviations

Top	for	Top(Ty)	
X :: K	for	X <: Top(K) :: K	(in environments and binders)
X <: A	for	X <: A :: Ty	(in environments and binders)
X	for	X <: Top :: Ty	(in environments and binders)
E ⊢ A	for	E ⊢ A :: Ty	
E ⊢ A :: K	for	E ⊢ A <: Top(K) :: K	
E ⊢ A <: B	for	E ⊢ A <: B :: Ty	
E ⊢ A ↔ B	for	E ⊢ A ↔ B :: Ty	

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Environment Formation ($E \vdash \diamond$)

(Env ∅)
$$\frac{}{\emptyset \vdash \diamond}$$

(Env X)
$$\frac{E \vdash A :: K \quad X \notin dom(E)}{E, X<:A::K \vdash \diamond}$$

(Env x)
$$\frac{E \vdash A \quad x \notin dom(E)}{E, x:A \vdash \diamond}$$

Kind Formation ($E \vdash K$ kind)

(Kind Ty)
$$\frac{E \vdash \diamond}{E \vdash Ty \text{ kind}}$$

(Kind Π)
$$\frac{E, X<:A::K \vdash L \text{ kind}}{E \vdash \Pi(X<:A::K)L \text{ kind}}$$

Kind Inclusion ($E \vdash K <:: L$ kind) [optional]

(Kind Sub Refl)
$$\frac{E \vdash K \leftrightarrow L \text{ kind}}{E \vdash K <:: L \text{ kind}}$$

(Kind Sub Trans)
$$\frac{E \vdash K <:: L \text{ kind} \quad E \vdash L <:: M \text{ kind}}{E \vdash K <:: M \text{ kind}}$$

(Kind Sub Ty)
$$\frac{E \vdash \diamond}{E \vdash Ty <:: Ty \text{ kind}}$$

(Kind Sub Π)
$$\frac{E \vdash A' <: A :: K \quad E, X<:A'::K \vdash L <:: L' \text{ kind}}{E \vdash \Pi(X<:A::K)L <:: \Pi(X<:A'::K)L' \text{ kind}}$$

Kind Equivalence ($E \vdash K \leftrightarrow L$ kind)

(Kind Eq Refl)
$$\frac{E \vdash K \leftrightarrow L \text{ kind}}{E \vdash L \leftrightarrow K \text{ kind}}$$

(Kind Eq Trans)
$$\frac{E \vdash K \leftrightarrow L \text{ kind} \quad E \vdash L \leftrightarrow M \text{ kind}}{E \vdash K \leftrightarrow M \text{ kind}}$$

(Kind Eq Ty)
$$\frac{E \vdash \diamond}{E \vdash Ty \leftrightarrow Ty \text{ kind}}$$

(Kind Eq Π)
$$\frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X<:A::K \vdash L \leftrightarrow L' \text{ kind}}{E \vdash \Pi(X<:A::K)L \leftrightarrow \Pi(X<:A'::K')L' \text{ kind}}$$

Constructor Formation ($E \vdash A :: K$)

(Con Extension)
$$\frac{E \vdash A :: K \quad E \vdash K \leftrightarrow L \text{ kind}}{E \vdash A :: L}$$

(Con Subsumption) [Replaces (Con Extension), with kind inclusion]
$$\frac{E \vdash A :: K \quad E \vdash K <:: L \text{ kind}}{E \vdash A :: L}$$

(Con X)
$$\frac{E', X<:A::K, E'' \vdash \diamond}{E', X<:A::K, E'' \vdash X :: K}$$

(Con Top)
$$\frac{E \vdash K \text{ kind}}{E \vdash Top(K) :: K}$$

(Con Abs)
$$\frac{E, X<:A::K \vdash B :: L}{E \vdash \lambda(X<:A::K)B :: \Pi(X<:A::K)L}$$

(Con Appl)
$$\frac{E \vdash B :: \Pi(X<:A::K)L\{X\} \quad E \vdash A'<:A::K}{E \vdash B(A') :: L\{A'\}}$$

$\mathcal{F}_{w<:}$

(Type Arrow)
$$\frac{E \vdash A \quad E \vdash B}{E \vdash A \to B}$$

(Type All)
$$\frac{E, X<:A::K \vdash B}{E \vdash \forall(X<:A::K)B}$$

(Type Rec)
$$\frac{E, X \vdash A \quad A > X}{E \vdash \mu(X)A}$$

Constructor Inclusion ($E \vdash A <: B :: K$)

(Con Sub Refl)
$$\frac{E \vdash A \leftrightarrow B :: K}{E \vdash A <: B :: K}$$

(Con Sub Trans)
$$\frac{E \vdash A <: B :: K \quad E \vdash B <: C :: K}{E \vdash A <: C :: K}$$

(Con Sub X)
$$\frac{E', X<:A::K, E'' \vdash \diamond}{E', X<:A::K, E'' \vdash X <: A :: K}$$

(Con Sub Top)
$$\frac{E \vdash A :: K}{E \vdash A <: Top(K) :: K}$$

(Con Sub Abs)
$$\frac{E, X<:A::K \vdash B <: B' :: L}{E \vdash \lambda(X<:A::K)B <: \lambda(X<:A::K)B :: \Pi(X<:A::K)L}$$

(Con Sub Appl)
$$\frac{E \vdash B <: B' :: \Pi(X<:A::K)L\{X\} \quad E \vdash A' <: A::K}{E \vdash B(A') <: B'(A') :: L\{A'\}}$$

(Sub Arrow)
$$\frac{E \vdash A' <: A \quad E \vdash B <: B'}{E \vdash A \to B <: A' \to B'}$$

(Sub All)
$$\frac{E \vdash A' <: A::K \quad E, X<:A'::K \vdash B <: B'}{E \vdash \forall(X<:A::K)B <: \forall(X<:A'::K)B'}$$

(Sub Rec)
$$\frac{E \vdash \mu(X)A \quad E \vdash \mu(Y)B \quad E, Y, X<:Y \vdash A <: B}{E \vdash \mu(X)A <: \mu(Y)B}$$

Constructor Equivalence ($E \vdash A \leftrightarrow B :: K$)

(Con Eq Symm)
$$\frac{E \vdash A \leftrightarrow B :: K}{E \vdash B \leftrightarrow A :: K}$$

(Con Eq Trans)
$$\frac{E \vdash A \leftrightarrow B :: K \quad E \vdash B \leftrightarrow C :: K}{E \vdash A \leftrightarrow C :: K}$$

(Con Eq X)
$$\frac{E \vdash X :: K}{E \vdash X \leftrightarrow X :: K}$$

(Con Eq Top)
$$\frac{E \vdash K \text{ kind}}{E \vdash Top(K) \leftrightarrow Top(K) :: K}$$

(Con Eq Abs)
$$\frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X<:A::K \vdash B \leftrightarrow B' :: L}{E \vdash \lambda(X<:A::K)B \leftrightarrow \lambda(X<:A'::K')B' :: \Pi(X<:A::K)L}$$

(Con Eq Appl)

$$\frac{E \vdash B \leftrightarrow B' :: \Pi(X<:A'' :: K)L\{X\} \quad E \vdash A <: A'' :: K \quad E \vdash A \leftrightarrow A' :: K}{E \vdash B(A) \leftrightarrow B'(A') :: L\{A\}}$$

(Type Eq All)

$$\frac{E \vdash A \leftrightarrow A' :: K \quad E, X<:A::K \vdash B \leftrightarrow B'}{E \vdash \forall(X<:A::K)B \leftrightarrow \forall(X<:A'::K)B'}$$

(Type Eq Arrow)

$$\frac{E \vdash A \leftrightarrow A' \quad E \vdash B \leftrightarrow B'}{E \vdash A \to B \leftrightarrow A' \to B' \text{ type}}$$

(Type Eq Rec)

$$\frac{E, X \vdash B \leftrightarrow B'}{E \vdash \mu(X)B \leftrightarrow \mu(X)B'}$$

(Con Eq Eta)

$$\frac{E \vdash B :: \Pi(X<:A::K)L \quad X \notin dom(E)}{E \vdash (\lambda(X<:A::K)B(X)) \leftrightarrow B :: \Pi(X<:A::K)L}$$

(Con Eq Beta)

$$\frac{E \vdash (\lambda(X<:A::K)B\{X\})(A') :: L \quad E \vdash A' <: A :: K}{E \vdash (\lambda(X<:A::K)B)(A) \leftrightarrow B\{A'\} :: L}$$

Value Formation ($E \vdash a : A$)

(Val Subsumption)

$$\frac{E \vdash a:A \quad E \vdash A<:B}{E \vdash a : B}$$

(Val x)

$$\frac{E', x:A, E'' \vdash \diamond}{E', x:A, E'' \vdash x : A}$$

(Val Abs)

$$\frac{E, x:A \vdash b : B}{E \vdash \lambda(x:A)b : A \to B}$$

(Val Appl)

$$\frac{E \vdash b : A \to B \quad E \vdash a : A}{E \vdash b(a) : B}$$

(Val Abs2)

$$\frac{E, X<:A::K \vdash b : B}{E \vdash \lambda(X<:A::K)b : \forall(X<:A::K)B}$$

(Val Appl2)

$$\frac{E \vdash b : \forall(X<:A::K)B\{X\} \quad E \vdash A'<:A::K}{E \vdash b(A') : B\{A'\}}$$

(Val Fold)

$$\frac{E \vdash a : B\{\mu(X)B\{X\}\}}{E \vdash fold(\mu(X)B\{X\}, a) : \mu(X)B\{X\}}$$

(Val Unfold)

$$\frac{E \vdash a : \mu(X)B\{X\}}{E \vdash unfold(a) : B\{\mu(X)B\{X\}\}}$$

(Val Rec)

$$\frac{E, x:A \vdash b:A}{E \vdash \mu(x:A)b : A}$$

Value equivalence ($E \vdash a \leftrightarrow b : A$)

(Val Eq Refl)

$$\frac{E \vdash a : A}{E \vdash a \leftrightarrow a : A}$$

(Val Eq Symm)

$$\frac{E \vdash a \leftrightarrow b : A}{E \vdash b \leftrightarrow a : A}$$

(Val Eq Trans)

$$\frac{E \vdash a \leftrightarrow b : A \quad E \vdash b \leftrightarrow c : A}{E \vdash a \leftrightarrow c : A}$$

$\mathcal{F}_{w<:}$

(Val Eq x)

$$\frac{E \vdash x : A}{E \vdash x \leftrightarrow x : A}$$

(Val Eq Top)

$$\frac{E \vdash a : Top \quad E \vdash b : Top}{E \vdash a \leftrightarrow b : Top}$$

(Val Eq Abs)

$$\frac{E \vdash A \leftrightarrow A' \quad E, x{:}A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(x{:}A)b \leftrightarrow \lambda(x{:}A')b' : A \to B}$$

(Val Eq Appl)

$$\frac{E \vdash b \leftrightarrow b' : A \to B \quad E \vdash a \leftrightarrow a' : A}{E \vdash b(a) \leftrightarrow b'(a') : B}$$

(Val Eq Abs2)

$$\frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X{<:}A::K \vdash b \leftrightarrow b' : B}{E \vdash \lambda(X{<:}A::K)b \leftrightarrow \lambda(X{<:}A'::K')b' : \forall(X{<:}A::K)B}$$

(Val Eq Appl2)

$$\frac{E \vdash b \leftrightarrow b' :: \forall(X{<:}A''::K)B\{X\} \quad E \vdash A <: A'' :: K \quad E \vdash A \leftrightarrow A' :: K}{E \vdash b(A) \leftrightarrow b'(A') : B\{A\}}$$

(Val Eq Fold)

$$\frac{E \vdash A \leftrightarrow \mu(X)B\{X\} \quad E \vdash A \leftrightarrow A' \quad E \vdash a \leftrightarrow a' : B\{A\}}{E \vdash fold(A,a) \leftrightarrow fold(A',a') : A}$$

(Val Eq Unfold)

$$\frac{E \vdash a \leftrightarrow a' : \mu(X)B\{X\}}{E \vdash unfold(a) \leftrightarrow unfold(a') : \mu(X)B\{X\}}$$

(Val Eq Rec)

$$\frac{E \vdash A \leftrightarrow A' \text{ type} \quad E, x{:}A \vdash b \leftrightarrow b' : A}{E \vdash \mu(x{:}A)b \leftrightarrow \mu(x{:}A')b' : A}$$

(Val Eq Eta2)

$$\frac{E \vdash b : \forall(X{<:}A::K)B \quad X \notin dom(E)}{E \vdash (\lambda(X{<:}A::K)b(X)) \leftrightarrow b : \forall(X{<:}A::K)B}$$

(Val Eq Eta)

$$\frac{E \vdash b : A \to B \quad x \notin dom(E)}{E \vdash (\lambda(x{:}A)b(x)) \leftrightarrow b : A \to B}$$

(Val Eq Beta2)

$$\frac{E \vdash (\lambda(X{<:}A::K)b\{X\})(A) : B}{E \vdash (\lambda(X{<:}A::K)b)(A) \leftrightarrow b\{A\} : B}$$

(Val Eq Beta)

$$\frac{E \vdash (\lambda(x{:}A)b\{x\})(a) : B}{E \vdash (\lambda(x{:}A)b)(a) \leftrightarrow b\{a\} : B}$$

(Val Eq Fold-Unfold)

$$\frac{E \vdash a : B\{\mu(X)B\{X\}\}}{E \vdash unfold(fold(\mu(X)B\{X\},a)) \leftrightarrow a : B\{\mu(X)B\{X\}\}}$$

(Val Eq Unfold-Fold)

$$\frac{E \vdash a : \mu(X)B\{X\}}{E \vdash fold(\mu(X)B\{X\},unfold(a)) \leftrightarrow a : \mu(X)B\{X\}}$$

(Val Eq Fix)

$$\frac{E \vdash \mu(x{:}A)b\{x\} : A}{E \vdash \mu(x{:}A)b \leftrightarrow b\{\mu(x{:}A)b\} : A}$$