

The Measurable Space of Stochastic Processes

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Motivation

Complex networks are often modelled as **stochastic processes**

- to encapsulate a lack of knowledge or inherent non-determinism,

E.g., natural systems (bio-chemical, ecological or physical)

- to model and analyse the global behaviours of systems containing components with both continuous-time and discrete-time evolutions,

E.g., embedded systems, communication networks, the Internet, service-oriented architectures, web-services, financial systems, market scenarios, etc.

Such systems are frequently **modular in nature**

- consist of modules which are systems in their own right,
- the modules interact, communicate and interrupt each other
- the global behaviour depends on the behaviours of the modules and on their links,
- the modules are easier to model, test, measure, analyse.

A possible approach: **Stochastic Process Algebras**

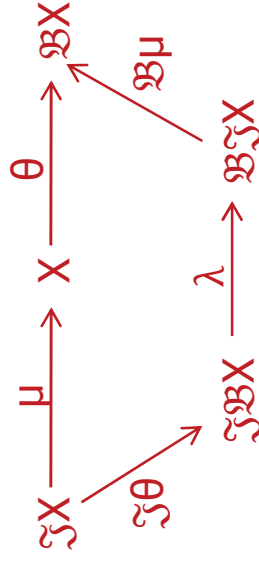
Nondeterministic Process Algebras

A successful solution for the nondeterministic case => **Process Algebras (PAs)**

- The concurrent communicating systems are conceptualised along two axes:
 - the behaviours are described by **(labelled) transition systems (LTS)** => **coalgebras**;
 - the compositionality is solved by **construction principles** => **algebraic structure**;
 - the algebraic and the coalgebraic structures are not independent:
 - structural operational semantics (SOS)** relates the two by defining the behaviour of a bigger system from the behaviours of its modules => **bialgebra**.

Two endofunctors \mathfrak{B} (for behavior) and \mathfrak{S} (for compositional specifications)

the class X of systems is simultaneously a **\mathfrak{B} -coalgebra** and a **\mathfrak{S} -algebra**.



- λ (a natural transformation between \mathfrak{S} and \mathfrak{B}) defines a **GSOS** \Rightarrow **$\mathfrak{S}\mathfrak{B}$ -Bialgebra**

D. Turi, G. Plotkin, *Towards a mathematical operational semantics*, LICS'97

Nondeterministic Process Algebras

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 - the compositionality is solved by **construction principles** => **algebraic structure**;
 - the algebraic and the coalgebraic structures are not independent:
 - structural operational semantics (SOS)** relates the two by defining the behaviour of a bigger system from the behaviours of its modules => **bialgebra**.
- PAs reflect elegant mathematical structures and are supported by easy and appealing underlying theories.
- PAs are based on a general framework for operational description of processes (GSOS) that makes the languages easy to extend with new (algebraic) operators.
- The underlying theory of PAs is robust: it can be canonically adapted when new algebraic operators are required => high applicability in various fields.

The challenge of stochastic processes

- The success of nondeterministic Process Algebras has inspired the research in the field of stochastic systems => **Stochastic Process Algebras (SPAs)**
- LTS is replaced by (labelled) Markovian processes (e.g., CTMCs)
 - nondeterministic a-transition:
$$P \xrightarrow{a} Q$$
 - stochastic (Markovian) a-transition:
$$P \xrightarrow{a,r} Q$$

$r \in [0, +\infty)$ is the rate of an exponentially distributed random variable that characterises the a-transitions from P to Q.
- In recent decades a plethora of SPAs appeared, such as
 - TIPP (Gotz, Herzog, Rettelbach)
 - PEPA (Hillston)
 - EMPA (Bernardo, Gorrieri)
 - Stochastic pi-calculus (Priami, Degano)
 - StoKlaim (De Nicola, Katoen, Latella, Loret, Massink)etc.

The challenge of stochastic processes

- In SPAs the nondeterminism is replaced by a “race policy” and this requires major modifications at the level of the underlying theory.
- The attempts to provide a pointwise semantics, similar to nondeterministic PAs, face the *counting problems* and the known SPAs solve them using “ad hoc” solutions such as the **multi-transition system** (PEPA) or the **proved SOS** (stochastic pi-calculus).

Problems

(B. Klin, V. Sassone, *Structural Operational Semantics for Stochastic Process Calculi*, FOSSACS’08)

- These SOS formalisms are difficult to extend to a general format for well-behaved stochastic specifications;
- In stochastic pi-calculus (with proved SOS) parallel composition is not associative up to bisimulation;
- In PEPA, if arbitrary relations between transition rates and the rates of subprocesses are allowed, stochastic bisimulation is not a congruence;

A possible explanation (*ibid.*): in a well-behaved SOS framework the labels of transitions should only carry as much data as required for the derivation of the intended semantics; Both the **proofs** and the **transition multiplicities** contain superfluous data.

The challenge of stochastic processes

A solution to return to the simplicity and elegance of nondeterministic PAs:

instead the **pointwise semantics**, use a **semantics based on distributions**.

$$P \xrightarrow{a,r} Q \quad \Rightarrow \quad P \longrightarrow \mu, \quad \mu(a)(\{Q\})=r$$

where μ is a distribution (indexed by actions) on the measurable space of processes.

Similar approaches

- R. Segala, N. Lynch, *Probabilistic Simulations for Probabilistic processes*, 1995.
- M. Kwiatkowska, G. Norman, R. Segala, J. Sproston, *Automatic Verification of Real-Time Systems with Discrete Probability Distributions*, 1999.
- E. P. de Vink, J. Rutten, *Bisimulation for probabilistic transition systems: A coalgebraic approach*, 1999.
- J. Rutten, *Universal Coalgebra: a theory of systems*, 2000.
- F. Bartels, *On Generalised Coinduction and Probabilistic Specification Formats*, 2004.
- M. Bravetti, H. Hermanns, J.-P. Katoen, *YMCA: Why Markov Chain Algebra?*, 2006.
- B. Klin, V. Sassone, *Structural Operational Semantics for Stochastic Process Calculi*, 2008.
- R. De Nicola, D. Latella, M. Loreti, M. Massink, *Rate-based Transition Systems for Stochastic Process Calculi*, 2009.

In general, the space of processes considered is $(\mathbf{P}, 2^{\mathbf{P}})$ – a compact representation of the pointwise semantics.

The role of Structural Congruence

Our approach: **Structural congruence organizes a measurable space of processes**; instead of $2^{\mathbf{P}}$ we use the sigma algebra Π generated by \mathbf{P}^{\equiv} ; we use distributions on (\mathbf{P}, Π) .

Some of the advantages of our approach:

- Structural congruence is particularly appropriate for applications in Systems Biology (it has been invented from chemical analogy G. Berry, G. Boudol, *The Chemical Abstract Machine*, 1990);
- By considering the distributions on (\mathbf{P}, Π) , the counting problems are solved on block while the labels of the transition are the observable actions, as in the case of nondeterministic PAs;
- SOS is elegant, compact and the algebraic and coalgebraic structures are clean: SOS does not involve a rule of type (Struct); stochastic bisimulation is a congruence and it extends the structural congruence;
- The approach can be extended to other calculi; a general format can be defined, on the line of GSOS (Turi-Plotkin) or SGSOS (Klin-Sassone);
- The approach, extended to pi-calculus, provides a simple solution to the problems of bound output and replication (Cardelli, Mardare, *Stochastic Pi-Calculus revisited*, <http://lucacardelli.name/>);
- The approach allows a simple extension to metric semantics.

Stochastic CCS: The syntax

Let \mathbf{A} be a denumerable set of action names endowed with

- an involution $*$: $\mathbf{A} \longrightarrow \mathbf{A}$, $a^* \neq a$, $a^{**} = a$
- a weight function $\iota: \mathbf{A} \longrightarrow \mathbb{Q}^+$, $\iota(a) = \iota(a^*)$ for all $a \in \mathbf{A}$

Let $\zeta \notin \mathbf{A}$ and $\mathbf{A}^+ = \mathbf{A} \cup \{\zeta\}$

The set \mathbf{P} of processes are defined by the following grammar, for arbitrary $r \in \mathbb{Q}^+$.

$$\mathbf{P} ::= 0 \mid \varepsilon.P \mid P|P \mid P+P$$

$$\varepsilon ::= a \in \mathbf{A} \mid \zeta(r)$$

We extend the weight function ι by $\iota(\zeta(r)) = r$.

Structural Congruence " \equiv " is the smallest equivalence relation on \mathbf{P} that satisfies

- I. 1. $P|Q \equiv Q|P$; 2. $(P|Q)|R \equiv P|(Q|R)$; 3. $P|0 \equiv P$.
- II. 1. $P+Q \equiv Q+P$; 2. $(P+Q)+R \equiv P+(Q+R)$; 3. $P+0 \equiv P$.
- III. if $P \equiv Q$, then for any ε and any $R \in \mathbf{P}$,
 1. $P|R \equiv Q|R$; 2. $P+R \equiv Q+R$;
 3. $\varepsilon.P \equiv \varepsilon.Q$.

The measurable space

For arbitrary $P \in \mathbf{P}$, let P^{\equiv} be the \equiv -equivalence class of P and \mathbf{P}^{\equiv} the set of \equiv -equivalence classes of processes.

Let Π be the sigma-algebra generated by \mathbf{P}^{\equiv} over \mathbf{P} .

Theorem: (\mathbf{P}, Π) is a measurable space.

Let $\Delta(\mathbf{P}, \Pi)$ denote the set of measurable functions on (\mathbf{P}, Π) .

For $S, T \in \Pi$, let $S | T = \bigcup_{P \in S, Q \in T} (P|Q)^{\equiv}$ and $S_T = \bigcup_{P \in S, P \in T} R^{\equiv}$

Lemma: If $S, T \in \Pi$ and $P \in \mathbf{P}$, then $S | T$ and S_T are measurable sets.

Structural Operational Semantics

First challenge: define the operational semantics based on rules of type

$$P \longrightarrow \mu,$$

where $\mu: \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi)$ is such that

for each $x \in \mathbf{A}^+$, $\mu(x) \in \Delta(\mathbf{P}, \Pi)$ is a measure and

for each $S \in \Pi$, $\mu(x)(S) = r \in \mathbb{Q}^+$,

r is the rate of the x -transition from P to (elements of) S .

Notice that S is not just any set, but a measurable set, e.g. $\mu(x)(\{Q\})$ is undefined.

More generally, the operational semantics encodes a function

$$\theta: \mathbf{P} \longrightarrow [\mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi)]$$

defined by $\theta(P) = \mu$ iff $P \longrightarrow \mu$.

In this way, a stochastic process $P \in \mathbf{P}$ is just a **Markov process** $(P, \mathbf{P}, \Pi, \theta)$, where $(\mathbf{P}, \Pi, \theta)$ is an **\mathbf{A}^+ - Markov kernel**.

Second challenge: define the operational semantics such that

behavioural equivalence of SPA processes = stochastic bisimulation of Markov processes

Structural Operational Semantics

The null process "0"

Intuition: for each measurable set $S \in \Pi$, and any action $x \in \mathbf{A}^+$,

$$0 \xrightarrow{x,0} S.$$

Let $\varpi : \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi)$ such that for any $x \in \mathbf{A}^+$,

$$\varpi(x) = \omega,$$

where $\omega \in \Delta(\mathbf{P}, \Pi)$ is the null measure defined, for arbitrary $S \in \Pi$, by

$$\omega(S) = 0.$$

The first SOS rule is

$$\text{(Null)} \quad \frac{}{0 \longrightarrow \varpi}$$

Structural Operational Semantics

The prefixing “ $\varepsilon.P$ ”

Intuition: $a.P \xrightarrow{a, \iota(a)} P^\equiv,$

for any measurable set $S \in \Pi$, with $P \notin S$,

$$a.P \xrightarrow{a, 0} S,$$

for any measurable set $S \in \Pi$ and any $x \neq a$,

$$a.P \xrightarrow{x, 0} S.$$

$$\zeta(r).P \xrightarrow{\zeta, r} P^\equiv,$$

for any measurable set $S \in \Pi$, with $P \notin S$,

$$\zeta(r).P \xrightarrow{\zeta, 0} S,$$

for any measurable set $S \in \Pi$ and any $x \neq \zeta$,

$$\zeta(r).P \xrightarrow{x, 0} S.$$

Let $[\varepsilon]_{P^\equiv} : \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi)$ such that for any $a \in \mathbf{A}$,

$$[\varepsilon]_{P^\equiv}(a) = \begin{cases} D(\iota(\varepsilon), P^\equiv), & a = \varepsilon \\ \omega, & a \neq \varepsilon \end{cases} \quad [\varepsilon]_{P^\equiv}(\zeta) = \begin{cases} D(\iota(\varepsilon), P^\equiv), & \varepsilon \notin \mathbf{A} \\ \omega, & \varepsilon \in \mathbf{A} \end{cases}$$

The second SOS rule is

$$\text{(Guard)} \quad \frac{}{\varepsilon.P \longrightarrow [\varepsilon]_{P^\equiv}}$$

Structural Operational Semantics

The nondeterministic choice “+”

Intuition: if for some measurable set $S \in \Pi$ and some action x

$$P \xrightarrow{x,r} S \quad \text{and} \quad Q \xrightarrow{x,s} S,$$

$$\text{then,} \quad P+Q \xrightarrow{x,r+s} S.$$

Let $\oplus : \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \times \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$

such that for any $\mu, \mu' \in \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$, any action x and any measurable set S ,

$$(\mu \oplus \mu')(x)(S) = \mu(x)(S) + \mu'(x)(S)$$

The third SOS rule is

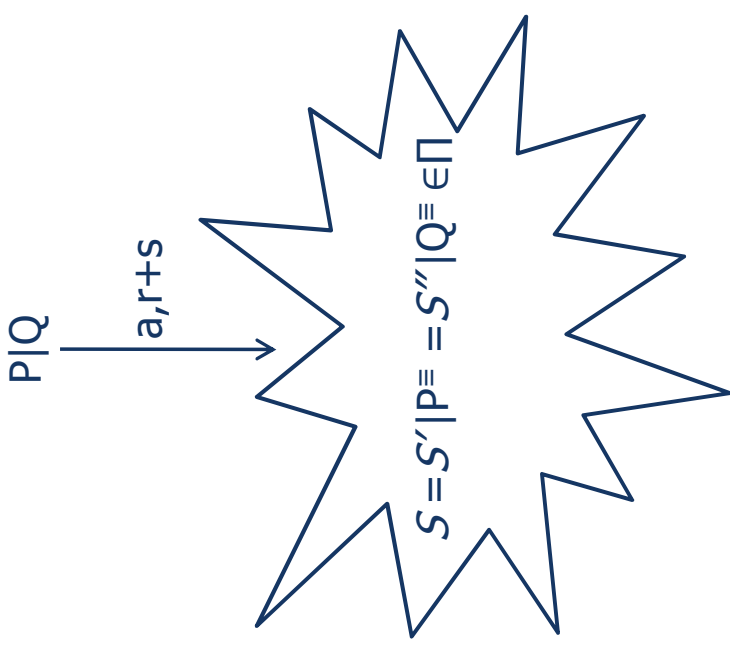
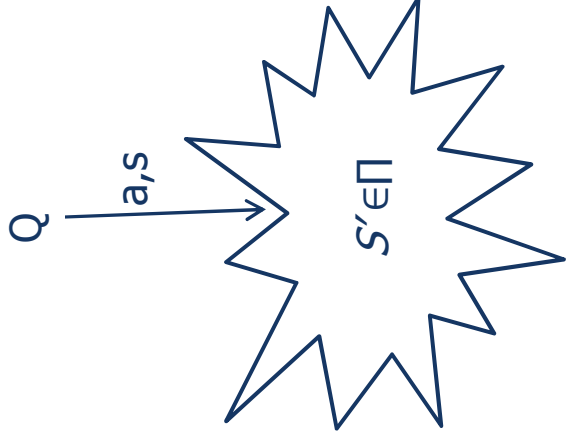
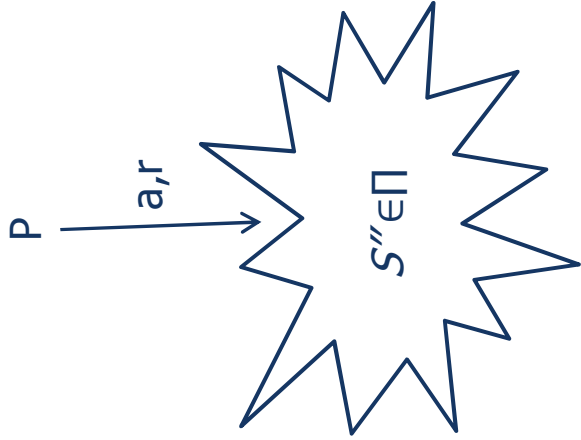
$$\text{(Sum)} \quad \frac{P \longrightarrow \mu \quad Q \longrightarrow \mu'}{P+Q \longrightarrow \mu \oplus \mu'}$$

Structural Operational Semantics

The parallel composition “|”

Intuition: for $a \in \mathbf{A}$, if $S = S' | P \equiv S'' | Q \equiv$ and $P \xrightarrow{a,r} S'$, $Q \xrightarrow{a,s} S''$

then $P | Q \xrightarrow{a,r+s} S$.



Let $\mu_{P \otimes Q} : \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \times \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$

such that for any $\mu, \mu' \in \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$, any atomic action a and any measurable set S ,

$$(\mu_{P \otimes Q} \mu')(a)(S) = \mu(a)(S_Q) + \mu'(a)(S_P)$$

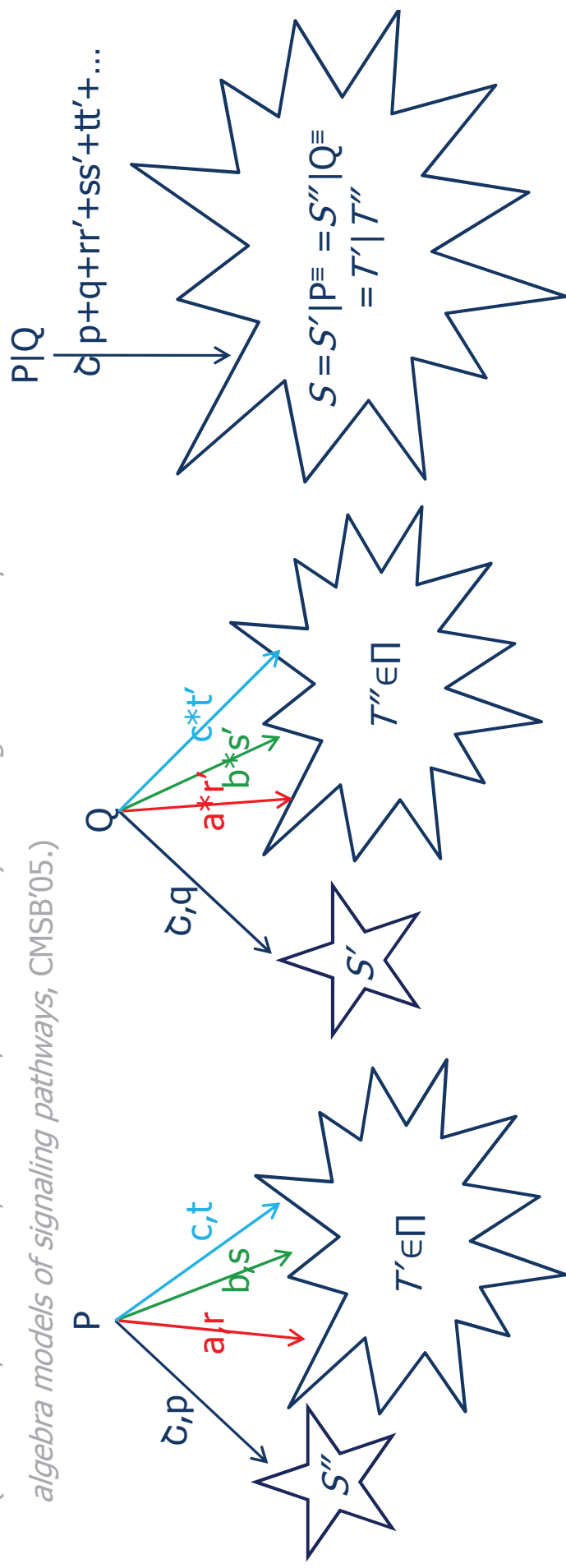
(if $P \longrightarrow \mu$ and $Q \longrightarrow \mu'$, then $P | Q \longrightarrow \mu_{P \otimes Q} \mu'$)

Structural Operational Semantics

Intuition: for $P|Q$ the rate of ζ -action subsumes

- the rate of ζ -action of P and the rate of ζ -action of Q
- the rate of synchronizations between P and Q - we use the **mass action law**.

(M. Calder, S. Gilmore, J. Hillston, *Automatically deriving ODEs from process algebra models of signaling pathways*, CMSB'05.)



for μ, μ' with finite support:

$$(\mu \otimes_Q \mu')(\zeta)(S) = \mu(\zeta)(S_Q) + \mu'(\zeta)(S_P) + \sum_{a \in A} \frac{\mu(a)(T') \times \mu'(a^*)(T'')}{2\iota(a)} \sum_{T' | T'' = S}$$

Structural Operational Semantics

The parallel composition “|”

For any $P, Q \in \Pi$, let $\mu \otimes_Q \mu' : \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \times \Delta(\mathbf{P}, \Pi) \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$ such that for any $\mu, \mu' \in \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$ (with finite support), any atomic action a and any $S \in \Pi$,

$$(\mu \otimes_Q \mu')(a)(S) = \mu(a)(S_Q) + \mu'(x)(S_P)$$

$$(\mu \otimes_Q \mu')(c)(S) = \mu(c)(S_Q) + \mu'(c)(S_P) + \sum_{\substack{a \in \mathbf{A} \\ T' | T'' = S}} \frac{\mu(a)(T') \times \mu'(a^*)(T'')}{2i(a)}$$

The fourth SOS rule is

$$\text{(Par)} \quad \frac{P \longrightarrow \mu \quad Q \longrightarrow \mu'}{P | Q \longrightarrow \mu \otimes_Q \mu'}$$

The algebra of mappings

We have defined an algebraic structure $(\Delta(\mathbf{P}, \Pi)^{\mathbf{A}^+}, \varpi, [\varepsilon]_P, \oplus, \rho \otimes_Q)$ with operators defined for arbitrary ε , P and Q .

Lemma:

- I. (1) $\mu \oplus \mu' = \mu' \oplus \mu$,
(2) $(\mu \oplus \mu') \oplus \mu'' = \mu \oplus (\mu' \oplus \mu'')$,
(3) $\mu \oplus \varpi = \mu$.
- II. (1) $\mu \rho \otimes_Q \mu' = \mu' \rho \otimes_Q \mu$,
(2) $(\mu \rho \otimes_Q \mu') \rho \otimes_Q \mu'' = \mu \rho \otimes_Q (\mu' \rho \otimes_Q \mu'')$,
(3) $\mu \rho \otimes_Q \varpi = \mu$.

Notice that $(\mathbf{P}, 0, \varepsilon, +, |)$ and $(\Delta(\mathbf{P}, \Pi)^{\mathbf{A}^+}, \varpi, [\varepsilon]_P, \oplus, \rho \otimes_Q)$ have different signatures.

Structural Operational Semantics

$$\text{(Null)} \quad \frac{}{0 \longrightarrow \varpi}$$

$$\text{(Guard)} \quad \frac{}{\varepsilon.P \longrightarrow [\varepsilon]_{P \equiv}}$$

$$\text{(Sum)} \quad \frac{P \longrightarrow \mu \quad Q \longrightarrow \mu'}{P+Q \longrightarrow \mu \oplus \mu'}$$

$$\text{(Par)} \quad \frac{P \longrightarrow \mu \quad Q \longrightarrow \mu'}{P|Q \longrightarrow \mu_P \otimes_Q \mu'}$$

Lemma: For any $P \in \mathbf{P}$, there exists a unique $\mu \in \Delta(\mathbf{P}, \Pi) \mathbf{A}^+$ such that $P \longrightarrow \mu$.
Moreover, μ has finite support.

Notice that we have no rule that guarantees that structural congruent processes have identical behaviour. But we can prove this.

Theorem: If $P \equiv Q$ and $P \longrightarrow \mu$, then $Q \longrightarrow \mu$.

Stochastic bisimulation

We can define $\theta: \mathbf{P} \longrightarrow [\mathbf{A}^+ \longrightarrow \Delta(\mathbf{P}, \Pi)]$ by $\theta(\mathbf{P}) = \mu$ iff $\mathbf{P} \longrightarrow \mu$.

Theorem: $(\mathbf{P}, \Pi, \theta)$ is an \mathbf{A}^+ -Markov kernel and $(\mathbf{P}, \Pi, \theta)$ is a Markov process.

Definition: A *rate bisimulation* is an equivalence \mathcal{R} on \mathbf{P} such that for arbitrary $\mathbf{P}, \mathbf{Q} \in \mathbf{P}$ with $\mathbf{P} \longrightarrow \mu$ and $\mathbf{Q} \longrightarrow \mu'$, $(\mathbf{P}, \mathbf{Q}) \in \mathcal{R}$ iff for any $S \in \Pi(\mathcal{R})$ and any $x \in \mathbf{A}^+$,

$$\mu(x)(S) = \mu'(x)(S).$$

The **stochastic bisimulation** is the reunion of rate bisimulations. Notation: $\mathbf{P} \sim \mathbf{Q}$.

For arbitrary $\mathbf{P} \in \mathbf{P}$, let $\mathbf{P} \sim$ be the \sim -equivalence class of \mathbf{P} and $\mathbf{P} \sim$ the set of \sim -equivalence classes of processes.

Examples (discussed in the paper)

1. $a, b \in \mathbf{A}$, $a^* \neq b$, $a.P|b.Q \sim a.(P|b.Q) + b.(a.P|Q)$

2. $r \neq s$, $\delta(r).P | \delta(s).Q \sim \delta(r).(P | \delta(s).Q) + \delta(s).(P | \delta(r).P|Q)$


Stochastic Bisimulation

Examples (discussed in the paper)

3. We have seen that for $a \neq b$, $a.0|b.0 \sim a.b.0+b.a.0 \Rightarrow S = a.0|b.0 \sim a.b.0+b.a.0$

Let $P = \zeta(r).(a.0|b.0) + \zeta(r).(a.b.0+b.a.0)$

$Q = \zeta(r).(a.0|b.0) + \zeta(r).(a.0|b.0)$

$R = \zeta(r).(a.b.0+b.a.0) + \zeta(r).(a.b.0+b.a.0)$

Observe that

$$P \xrightarrow{\zeta, r} S$$

$$Q \xrightarrow{\zeta, 2r} S$$

$$R \xrightarrow{\zeta, 2r} S$$

and $P \sim Q \sim R$.

the three processes do not agree on any transition:

$$P \xrightarrow{\zeta, r} a.0|b.0$$

$$Q \xrightarrow{\zeta, 2r} a.0|b.0$$

$$R \xrightarrow{\zeta, 0} a.0|b.0$$

$$P \xrightarrow{\zeta, r} a.b.0+b.a.0$$

$$Q \xrightarrow{\zeta, 0} a.b.0+b.a.0$$

$$R \xrightarrow{\zeta, 2r} a.b.0+b.a.0$$

Stochastic Bisimulation

Lemma:

For arbitrary $P, Q \in \mathbf{P}$, if $P \equiv Q$, then $P \sim Q$.

The reverse is not true, as shown in a previous example:

$a.0|b.0 \sim a.b.0+b.a.0$ and $a.0|b.0 \not\equiv a.b.0+b.a.0$.

Theorem:

Stochastic bisimulation is a congruence, i.e.,

1. if $P \sim P'$, then for arbitrary ε , $\varepsilon.P \sim \varepsilon.P'$;
2. if $P \sim P'$ and $Q \sim Q'$, then $P+P' \sim Q+Q'$;
3. if $P \sim P'$ and $Q \sim Q'$, then $P|P' \sim Q|Q'$.

An Application: Metrics for Stochastic Processes

Bisimulation is a strict concept: it only verifies if two processes have identical behaviour.

It is useful to have a metric that measure the similarity of processes in terms of behaviours.

Our presentation of stochastic processes is particularly appropriate to define such a metric via **metrics for distributions (e.g. Kantorovich metrics)**:

The intuition: the distance between P and Q , when $P \longrightarrow \mu$ and $Q \longrightarrow \mu'$, is given by

$$d(P,Q) = \sup_{x \in A^+} \delta(\mu(x), \mu'(x))$$

where δ is a distance on distributions.

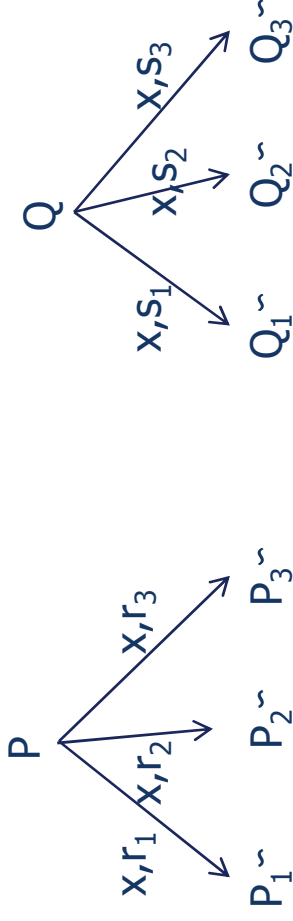
Related works on metrics for systems:

- P. Lincoln, J. Mitchell, M. Mitchell, A. Scedrov, *A probabilistic poly-time framework for protocol analysis*, 1998
- J. Desharnais, *Labelled Markov Processes*, 1999
- F. Van Breugel, J. Worrell, *An Algorithm for Quantitative Verification of Probabilistic Systems*, 2001
- L. de Alfaro, T. Henzinger, R. Majumdar, *Discounting the Future in Systems Theory*, 2003
- V. Gupta, R. Jagadeesan, P. Panangaden, *Approximate Reasoning for Real-Time Probabilistic Processes*, 2006

An Application: Metrics for Stochastic Processes

A discount metric: let $c \in [0,1]$ and $x \in \mathbb{A}^+$; we define the pseudometric

$$d_x^c : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{R}^+$$



$$d_x^c(P,Q) = \min \left\{ \begin{array}{l} |r_1 - s_1| + c d_x^c(P_1, Q_1) + |r_2 - s_2| + c d_x^c(P_2, Q_2) + |r_3 - s_3| + c d_x^c(P_3, Q_3), \\ |r_1 - s_1| + c d_x^c(P_1, Q_1) + |r_2 - s_3| + c d_x^c(P_2, Q_3) + |r_3 - s_2| + c d_x^c(P_3, Q_2), \\ |r_1 - s_2| + c d_x^c(P_1, Q_2) + |r_2 - s_1| + c d_x^c(P_2, Q_1) + |r_3 - s_3| + c d_x^c(P_3, Q_3), \\ |r_1 - s_2| + c d_x^c(P_1, Q_2) + |r_2 - s_3| + c d_x^c(P_2, Q_3) + |r_3 - s_1| + c d_x^c(P_3, Q_1), \\ |r_1 - s_3| + c d_x^c(P_1, Q_3) + |r_2 - s_1| + c d_x^c(P_2, Q_1) + |r_3 - s_2| + c d_x^c(P_3, Q_2), \\ |r_1 - s_3| + c d_x^c(P_1, Q_3) + |r_2 - s_2| + c d_x^c(P_2, Q_2) + |r_3 - s_1| + c d_x^c(P_3, Q_1) \end{array} \right\}$$

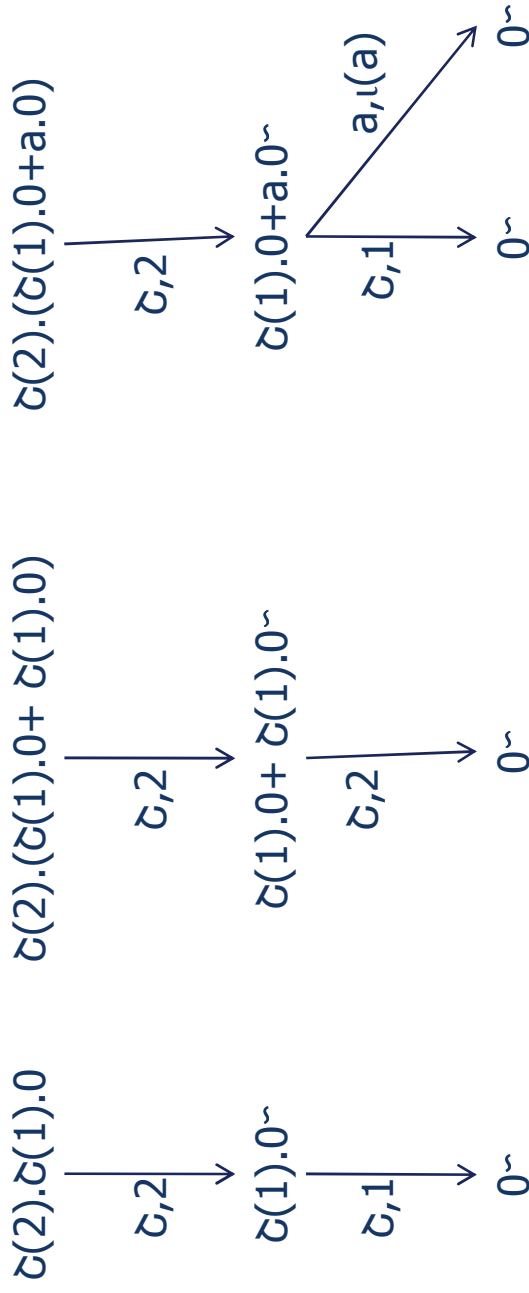
$$d^c : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{R}^+$$

$$d^c(P,Q) = \sup_{x \in \mathbb{A}^+} d_x^c(P,Q)$$

Notice that c discounts the future; if we take $c=1$ the future states count as the present state; if we take $c=0$ only the first step of the computation is considered.

An Application: Metrics for Stochastic Processes

Example (discussed in the paper)



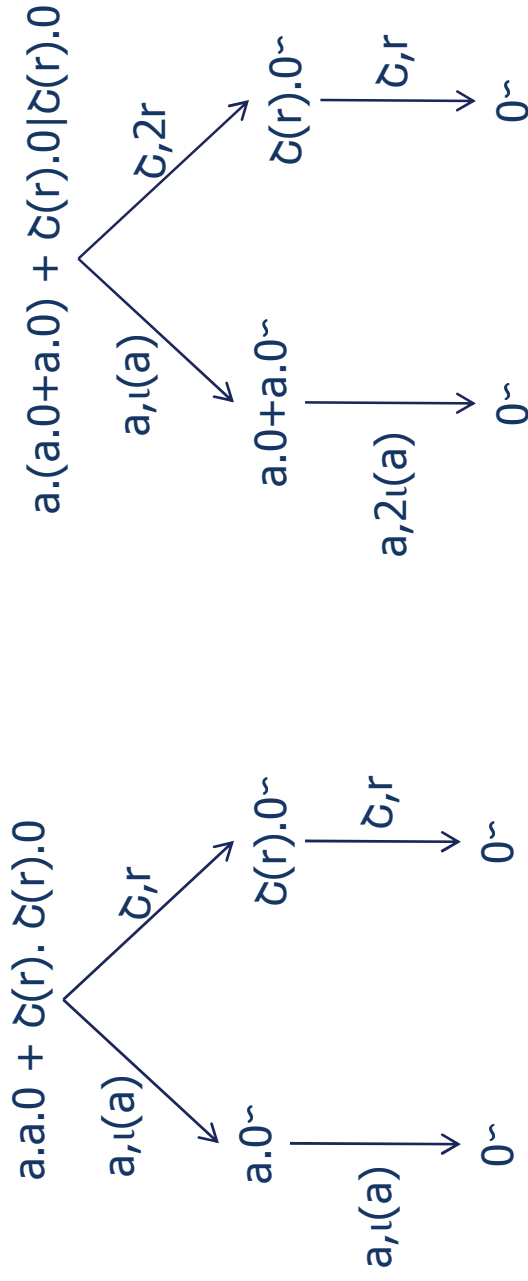
$$d^c_s(c(2).c(1).0, c(2).c(1).0+c(1).0) = |2-2| + c |2-1| = c$$

$$d^c_s(c(2).c(1).0, c(2).c(1).0+a.0) = |2-2| + c |1-1| = 0$$

$$d^c_s(c(2).c(1).0+c(1).0, c(2).c(1).0+a.0) = |2-2| + c |2-1| = c$$

An Application: Metrics for Stochastic Processes

Example (discussed in the paper)



$$d^c(a.a.0 + c(r).c(r).0, a.(a.0+a.0) + c(r).0|c(r).0) = \max \{c_1(a), r\}$$

Conclusions

- We took the challenge of reconsidering Stochastic Process Algebras from a foundational perspective
 - The goals:
 - understanding if the “ad hoc” approaches with their heavy mathematics can be avoided
 - providing well-behaved SOS formats similar to the formats of nondeterministic PAs
 - The way to do it:
 - instead of trying to mimic the pointwise semantics of PAs, mimic their mathematical structures – move from the space of processes to the space of distributions
 - center the work on the equational theory of structural congruence \Rightarrow work with the equational monad instead of the freely generated monad
 - lift the algebraic structure from the space of processes to the space of distributions
 - Advantages:
 - an elegant and compact SOS
 - well-behaved SOS: bisimulation is a congruence that extends structural congruence
 - a simple extension to metric semantics
 - simple solutions to the problems related to recursion and bound output
 - The current state of our research:
- L. Cardelli, R. Mardare, *Stochastic Pi-Calculus Revisited*, <http://lucacardelli.name/>**