By induction hypothesis, since $E \vdash a : [l_j{:}B_j, \ldots] \wedge \sigma{\bullet}S \vdash a \rightsquigarrow [l_i{=}\iota_i\ ^{i\in 1..n}]{\bullet}\sigma' \wedge \Sigma \vDash \sigma \wedge dom(\sigma) = dom(\Sigma) \wedge \Sigma \vDash S : E$, there exist a type A' and a store type Σ' such that $\Sigma' \geqslant \Sigma \wedge \Sigma' \vDash \sigma' \wedge dom(\sigma') = dom(\Sigma') \wedge \Sigma' \vDash [l_i{=}\iota_i\ ^{i\in 1..n}] : A' \wedge A' <: [l_j{:}B_j, \ldots]$.

Since $\sigma'(\iota_j) = \langle \varsigma(x_j)b_j, S' \rangle$, the judgment $\Sigma' \vDash \sigma'$ must come via (Store Typing) from $\Sigma' \vDash S' : E_j$ and $E_j, x_j{:}\Sigma'_1(\iota_j) \vdash b_j : \Sigma'_2(\iota_j)$ for some E_j. Since $\Sigma' \vDash [l_i{=}\iota_i\ ^{i\in 1..n}] : A'$ must come from (Result Object), we have $A' \equiv [l_i{:}\Sigma'_2(\iota_i)\ ^{i\in 1..n}] \equiv \Sigma'_1(\iota_j)$. Since $A' <: [l_j{:}B_j, \ldots]$, we have $\Sigma'_2(\iota_j) \equiv B_j$. Then from $E_j, x_j{:}\Sigma'_1(\iota_j) \vdash b_j : \Sigma'_2(\iota_j)$ we obtain $E_j, x_j{:}A' \vdash b_j : B_j$. Moreover, we get $\Sigma' \vDash S', x_j{\mapsto}[l_i{=}\iota_i\ ^{i\in 1..n}] : E_j, x_j{:}A'$ by the (Stack x Typing) rule.

Let $E' \equiv E_j, x_j{:}A'$. By induction hypothesis, since $E' \vdash b_j : B_j \wedge \sigma'{\bullet}(S', x_j{\mapsto}[l_i{=}\iota_i\ ^{i\in 1..n}]) \vdash b_j \rightsquigarrow v{\bullet}\sigma'' \wedge \Sigma' \vDash \sigma' \wedge dom(\sigma') = dom(\Sigma') \wedge \Sigma' \vDash S', x_j{\mapsto}[l_i{=}\iota_i\ ^{i\in 1..n}] : E'$, there exist A^+ and Σ^+ such that $\Sigma^+ \geqslant \Sigma' \wedge \Sigma^+ \vDash \sigma'' \wedge dom(\sigma'') = dom(\Sigma^+) \wedge \Sigma^+ \vDash v : A^+ \wedge A^+ <: B_j$.

We conclude:

- $\Sigma^+ \geqslant \Sigma$ by transitivity from $\Sigma^+ \geqslant \Sigma'$ and $\Sigma' \geqslant \Sigma$.
- $\Sigma^+ \vDash \sigma''$ with $dom(\sigma'') = dom(\Sigma^+)$.
- $\Sigma^+ \vDash v : A^+$ with $A^+ <: A$, by transitivity from $A^+ <: B_j$ and $B_j <: A$.

Case (Red Update)

$$\frac{\sigma{\bullet}S \vdash a \rightsquigarrow [l_i{=}\iota_i\ ^{i\in 1..n}]{\bullet}\sigma' \qquad j{\in}1..n \qquad \iota_j{\in}dom(\sigma')}{\sigma{\bullet}S \vdash a.l_j{\Leftarrow}\varsigma(x)b \rightsquigarrow [l_i{=}\iota_i\ ^{i\in 1..n}]{\bullet}(\sigma'.\iota_j{\leftarrowtail}\langle \varsigma(x)b, S\rangle)}$$

By hypothesis $E \vdash a.l_j{\Leftarrow}\varsigma(x)b : A \wedge \Sigma \vDash \sigma \wedge dom(\sigma) = dom(\Sigma) \wedge \Sigma \vDash S : E$. Since $E \vdash a.l_j{\Leftarrow}\varsigma(x)b : A$, we must have $E \vdash a : [l_j{:}B_j, \ldots]$ and $E, x{:}[l_j{:}B_j, \ldots] \vdash b : B_j$ for some $[l_j{:}B_j, \ldots] <: A$.

By induction hypothesis, since $E \vdash a : [l_j{:}B_j, \ldots] \wedge \sigma{\bullet}S \vdash a \rightsquigarrow [l_i{=}\iota_i\ ^{i\in 1..n}]{\bullet}\sigma' \wedge \Sigma \vDash \sigma \wedge dom(\sigma) = dom(\Sigma) \wedge \Sigma \vDash S : E$, there exist a type A^+ and a store type Σ^+ such that $\Sigma^+ \geqslant \Sigma \wedge \Sigma^+ \vDash \sigma' \wedge dom(\sigma') = dom(\Sigma^+) \wedge \Sigma^+ \vDash [l_i{=}\iota_i\ ^{i\in 1..n}] : A^+ \wedge A^+ <: [l_j{:}B_j, \ldots]$.

By assumption $\iota_j{\in}dom(\sigma')$, hence $\iota_j{\in}dom(\Sigma^+)$. But $\Sigma^+ \vDash [l_i{=}\iota_i\ ^{i\in 1..n}] : A^+$ must have been derived via (Result Object) from $\Sigma^+_1(\iota_i) \equiv [l_i{:}\Sigma^+_2(\iota_i)\ ^{i\in 1..n}] \equiv A^+$ for all $i{\in}1..n$. Hence, since $A^+ <: [l_j{:}B_j, \ldots]$, we have $\Sigma^+_2(\iota_j) = B_j$.

(1) We have $\Sigma^+ \vDash S : E$ by Lemma 11.4-1. We also have $E, x{:}A^+ \vdash b : B_j$ by Lemma 11.4-2 (from $E, x{:}[l_j{:}B_j, \ldots] \vdash b : B_j$ and $A^+ <: [l_j{:}B_j, \ldots]$), that is, $E, x{:}\Sigma^+_1(\iota_j) \vdash b : \Sigma^+_2(\iota_j)$.

(2) Since $\Sigma^+ \vDash \sigma'$ must come from (Store Typing), σ' is of the form $\varepsilon_k{\mapsto}\langle \varsigma(x_k)b_k, S_k\rangle\ ^{k\in 1..m}$, and for all k such that $\varepsilon_k \neq \iota_j$ and for some E_k we have $\Sigma^+ \vDash S_k : E_k$, and $E_k, x_k{:}\Sigma^+_1(\varepsilon_k) \vdash b_k : \Sigma^+_2(\varepsilon_k)$.

Take $\sigma^+ \equiv \sigma'.\iota_j{\leftarrowtail}\langle \varsigma(x)b, S\rangle$. Then by (1) and (2) we have $\Sigma^+ \vDash \sigma^+$, by the (Store Typing) rule. Since $dom(\sigma') = dom(\sigma^+)$, we have also $dom(\sigma^+) = dom(\Sigma^+)$.

We conclude $\Sigma^+ \geqslant \Sigma \wedge \Sigma^+ \vDash \sigma^+ \wedge dom(\sigma^+) = dom(\Sigma^+) \wedge \Sigma^+ \vDash [l_i{=}\iota_i\ ^{i\in 1..n}] : A^+ \wedge A^+ <: A$, with the last conjunct derived from $A^+ <: [l_j{:}B_j, \ldots]$ and $[l_j{:}B_j, \ldots] <: A$ by transitivity.